O176. Let $P(n)$ be the following statement: for all positive real numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n = n$,

$$\frac{x_2}{\sqrt{x_1 + 2x_3}} + \frac{x_3}{\sqrt{x_2 + 2x_4}} + \cdots + \frac{x_1}{\sqrt{x_n + 2x_2}} \ge \frac{n}{\sqrt{3}}.$$

Prove that $P(n)$ is true for $n \le 4$ and false for $n \ge 9$.

Proposed by Gabriel Dospinescu, Ecole Normale Superieure, France

First solution by the author

Let $S(x_1, x_2, \ldots, x_n)$ be the left hand side of the inequality. Using Holder's inequality, we obtain

$$S^2(x_2(x_1 + 2x_3) + \cdots + x_1(x_n + 2x_2)) \ge (x_1 + x_2 + \cdots + x_n)^3 = n^3.$$

On the other hand, we have

$$x_2(x_1 + 2x_3) + \cdots + x_1(x_n + 2x_2) = 3(x_1x_2 + x_2x_3 + \cdots + x_nx_1).$$

Using the fact that

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \le n$$

whenever $x_1 + x_2 + \cdots + x_n = n$ and $n \le 4$. The last fact follows from the fact that

$$ab + bc + ca \le \frac{(a+b+c)^2}{3}$$

and

$$ab + bc + cd + da = (a+c)(b+d) \le \frac{(a+b+c+d)^2}{4}.$$

The conclusion follows easily for $n \le 4$. Chosing x_1, x_2, x_3, x_4 close to $\frac{n}{4}$ and the other variables equal and close to 0, one easily obtains that the expression is smaller than $\frac{n}{\sqrt{3}}$ for $n \ge 9$. The conclusion follows.

Second solution by Arkady Alt, San Jose, California, USA

Let $n \le 4$.Then, applying consequentially AM-GM and Cauchy inequalities, we obtain

$$\sum_{cyc}^n \frac{x_2}{\sqrt{3(x_1 + 2x_3)}} \ge \sum_{cyc}^n \frac{2x_2}{3 + (x_1 + 2x_3)} = 2\sum_{cyc}^n \frac{x_2^2}{3x_2 + (x_1x_2 + 2x_2x_3)} \ge$$
$$\frac{2\left(\sum_{k=1}^n x_k\right)^2}{3\sum_{k=1}^n x_k + \sum_{cyc}^n (x_1x_2 + 2x_2x_3)} = \frac{2n^2}{3n + 3\sum_{cyc}^n x_1x_2}.$$

Thus, $\sum_{cyc}^n \frac{x_2}{\sqrt{x_1 + 2x_3}} \ge \frac{2}{\sqrt{3}} \cdot \frac{n^2}{n + \sum_{cyc}^n x_1x_2}$.

For $n = 3,$ since $x_1 + x_2 + x_3 = 3$ we have $\sum_{cyc}^n x_1x_2 \le \frac{(x_1 + x_2 + x_3)^2}{3} = 3.$

Then $\dfrac{2}{\sqrt{3}} \cdot \dfrac{n^2}{n+\sum_{cyc}^{n} x_1x_2} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{9}{3+\sum_{cyc}^{n} x_1x_2} \geq \dfrac{2}{\sqrt{3}} \cdot \dfrac{9}{6} = \sqrt{3} = \dfrac{3}{\sqrt{3}}$.

If $n=4$ then $x_1+x_2+x_3+x_4=4$ and $\sum_{cyc}^{n} x_1x_2 = (x_1+x_3)(x_2+x_4) \leq \left(\dfrac{(x_1+x_3)+(x_2+x_4)}{2}\right)^2 = 4$.Therefore, $\dfrac{2}{\sqrt{3}} \cdot \dfrac{n^2}{n+\sum_{cyc}^{n} x_1x_2} =$ $\dfrac{2}{\sqrt{3}} \cdot \dfrac{16}{4+\sum_{cyc}^{n} x_1x_2} \geq \dfrac{2}{\sqrt{3}} \cdot \dfrac{16}{4+4} = \dfrac{4}{\sqrt{3}}$.

Let $n \geq 9$ and let $x_k = \dfrac{n}{2^k}, k=1,2,...,n$.Then

$$L.H.S. = \sum_{k=1}^{n-2} \frac{x_{k+1}}{\sqrt{x_k+2x_{k+2}}} + \frac{x_n}{\sqrt{x_{n-1}+2x_1}} + \frac{x_1}{\sqrt{x_n+2x_2}} = \sum_{k=1}^{n-2} \frac{\frac{n}{2^{k+1}}}{\sqrt{\frac{n}{2^k}+2\cdot\frac{n}{2^{k+2}}}} +$$

$$\frac{\frac{n}{2^n}}{\sqrt{\frac{n}{2^{n-1}}+2\cdot\frac{n}{2}}} + \frac{\frac{n}{2}}{\sqrt{\frac{n}{2^n}+2\cdot\frac{n}{4}}} = \sum_{k=1}^{n-2} \frac{\sqrt{n}}{\sqrt{2^{k+2}+2^{k+1}}} + \frac{\sqrt{n}}{\sqrt{2^{n+1}+2^{2n}}} + \frac{\sqrt{n}}{\sqrt{\frac{1}{2^{n-2}}+2}}.$$

Since $\sum_{k=1}^{n-2} \dfrac{\sqrt{n}}{\sqrt{2^{k+2}+2^{k+1}}} = \sqrt{n}\sum_{k=1}^{n-2} \dfrac{1}{\sqrt{3\cdot 2^{k+1}}} = \sqrt{\dfrac{n}{3}}\sum_{k=1}^{n-2} \dfrac{1}{2\sqrt{2^{k-1}}} <$ $\dfrac{1}{2}\sqrt{\dfrac{n}{3}} \cdot \dfrac{1}{1-\dfrac{1}{\sqrt{2}}} = \dfrac{1}{2}\sqrt{\dfrac{n}{3}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}-1} = \sqrt{\dfrac{n}{6}}\left(\sqrt{2}+1\right),\ \dfrac{\sqrt{n}}{\sqrt{2^{n+1}+2^{2n}}} < \dfrac{\sqrt{n}}{2\sqrt{2^n}}$

and $\dfrac{\sqrt{n}}{\sqrt{\dfrac{1}{2^{n-2}}+2}} < \sqrt{\dfrac{n}{2}}$ then $L.H.S. < \sqrt{\dfrac{n}{3}}\left(\dfrac{\sqrt{2}+1}{\sqrt{2}} + \dfrac{\sqrt{3}}{2\sqrt{2^n}} + \dfrac{\sqrt{3}}{\sqrt{2}}\right)$.

Moreover, since $n \geq 9$ we obtain

$$1+\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2^n}}+\frac{\sqrt{3}}{\sqrt{2}} < \frac{\sqrt{2}+1}{\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2^9}}+\frac{\sqrt{3}}{\sqrt{2}} < 1+\frac{1+\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2^9}} = 2.976 < 3,$$

and , therefore, $L.H.S. < \sqrt{3n} < \dfrac{n}{\sqrt{3}}$.

So, $P(n)$ is false for $n \geq 9$.