O165. Let R and r be the circumradius and the inradius of a triangle ABC with the lengths of sides a, b, c. Prove that

$$2-2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \le \frac{r}{R}.$$

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Note that $2-2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \le \frac{r}{R} \iff 6-2\sum_{cyc}\left(\frac{a}{b+c}\right)^2 \le 4+\frac{r}{R} \iff$

$2\left(3-\sum_{cyc}\left(\frac{a}{b+c}\right)^2\right) \le 4+\frac{r}{R} \iff 2\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2} \le 4+\frac{r}{R}.$

Since $\cos A+\cos B+\cos C = 1+\frac{r}{R}$ and $\frac{1}{(b+c)^2} \le \frac{1}{4bc}$ then $\frac{(b+c)^2-a^2}{2bc} = 1+\cos A$

$2\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2} \le \sum_{cyc}\frac{(b+c)^2-a^2}{2bc} = \sum_{cyc}(1+\cos A) = 4+\frac{r}{R}.$

Remark.

Let l_a, l_b, l_c be angle bisectors of a triangle ABC.Noting that $\frac{(b+c)^2-a^2}{(b+c)^2} = \frac{al_a^2}{abc}$ we can rewrite original inequality in such form $2\sum_{cyc}\frac{al_a^2}{abc} \le 4+\frac{r}{R} \iff 2\sum_{cyc}\frac{al_a^2}{4Rrs} \le 4+\frac{r}{R} \iff$

$$\frac{al_a^2+bl_b^2+cl_c^2}{a+b+c} \le r\left(4R+r\right).$$

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