O161. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Magkos Athanasios, Kozani, Greece

We will make use of the following lemma which was proven in the solution of problem J131: If $x, y, z, a, b, c > 0$ we have

$$\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} \geq \frac{(x+y+z)^3}{(a+b+c)^2}.$$

We will also relax the condition $abc = 1$ to $abc \leq 1$. Set

$$a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}.$$

Then, we have $xyz \geq 1$ and the left hand side of the inequality is equal to

$$K = (xyz)^2 \sum_{cyc} \frac{x^3}{(2y+z)^2}.$$

From the above lemma (and since $xyz \geq 1$)we have

$$K \geq \frac{(x+y+z)^3}{9(x+y+z)^2} = \frac{x+y+z}{9} \geq \frac{3\sqrt[3]{xyz}}{9} \geq \frac{1}{3}.$$

Second solution by Arkady Alt, San Jose, California, USA

Since

$$\frac{1}{a^5\left(b+2c\right)^2} = \frac{1}{a^5b^2c^2\left(\dfrac{1}{c}+\dfrac{2}{b}\right)^2} = \frac{\left(\dfrac{1}{a}\right)^3}{\left(\dfrac{1}{c}+\dfrac{2}{b}\right)^2}$$

then by replacing $\left(\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}\right)$ in original inequality with (a, b, c) we obtain the equivalent inequality

$$\sum_{cyc} \frac{a^3}{\left(2b+c\right)^2} \geq \frac{1}{3},$$

with $abc = 1$. Note that $\dfrac{a^2}{2b+c} \geq \dfrac{2}{3}a - \dfrac{2b+c}{9}$; hence

$$\begin{aligned}\sum_{cyc} \frac{a^3}{(2b+c)^2} &\geq \sum_{cyc} \frac{a}{2b+c}\left(\frac{2}{3}a - \frac{2b+c}{9}\right) = \frac{2}{3}\sum_{cyc} \frac{a^2}{2b+c} - \sum_{cyc} \frac{a}{9} \\ &\geq \frac{2}{3}\sum_{cyc}\left(\frac{2}{3}a - \frac{2b+c}{9}\right) - \sum_{cyc}\frac{a}{9} = \frac{a+b+c}{3} - \frac{2}{9}\sum_{cyc}(2b+c) \\ &= \frac{a+b+c}{9} \geq \frac{3\sqrt[3]{abc}}{9} = \frac{1}{3}.\end{aligned}$$

Also solved by Daniel Lasaosa, Universidad Pública de Navarra, Spain; G. C. Greubel, USA;

Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy.