O150. Let $n$ be a positive integer, $\epsilon_0, \ldots, \epsilon_{n-1}$ be the $n$th roots of unity, and $a, b$ complex numbers. Evaluate the product

$$\prod_{k=0}^{n-1} \left(a + b\epsilon_k^2\right).$$

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*First solution by Carlo Pagano, Università di Roma "Tor Vergata", Roma, Italy*

1) If $n$ is odd then multiplication by 2 acts as a permutation on $\mathbb{Z}_n$ and therefore

$$P = \prod_{i=0}^{n-1} (a + b\epsilon_i) = \sum_{i=0}^{n-1} a^{n-i} b^i \sigma_i.$$

where $\sigma_i = \sum_{0 \le j_1 < \cdots < j_i \le n-1} \epsilon_{j_1} \cdots \epsilon_{j_i}$. Since

$$\sigma_0 = 1, \quad \sigma_{n-1} = (-1)^{n-1}, \quad \sigma_j = 0 \text{ for } j = 1, \ldots, n-2$$

we have that $P = a^n + b^n$ for $n$ odd.

2) If $n$ is even, since $\epsilon_{n/2+i} = -\epsilon_i$ then

$$P = \left(\prod_{i=0}^{n/2-1} (a + b(\epsilon_i)^2)\right)^2 = \left(\prod_{i=0}^{n/2-1} (a + b\omega_i)\right)^2$$

where $\omega_0, \ldots, \omega_{n/2-1}$ are the $n/2$-th roots of the unity. Hence, as before we find that $P = (a^{n/2} - (-b)^{n/2})^2$ for $n$ even.

*Second solution by Arkady Alt, San Jose, California, USA*

From the given we know that $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$ are roots of equation $u^n - 1 = 0, u \in \mathbb{C}$ then

$$\prod_{k=1}^{n-1} (u - \varepsilon_k) = z^n - 1.$$

Let $z$ be a solution of the equation $z^2 = -\frac{a}{b}$ then

$$\begin{aligned}\prod_{k=1}^{n-1} \left(a + b\varepsilon_k^2\right) &= \prod_{k=1}^{n-1} (-b) \left(\frac{a}{-b} - \varepsilon_k^2\right) = (-b)^n \prod_{k=1}^{n-1} \left(z^2 - \varepsilon_k^2\right) \\ &= (-b)^n \prod_{k=1}^{n-1} (z - \varepsilon_k) \prod_{k=1}^{n-1} (z + \varepsilon_k) = b^n \prod_{k=1}^{n-1} (z - \varepsilon_k) \prod_{k=1}^{n-1} ((-z) - \varepsilon_k) \\ &= b^n \left(z^n - 1\right) \left((-z)^n - 1\right) = b^n \left((-1)^n z^{2n} - z^n \left((-1)^n + 1\right) + 1\right).\end{aligned}$$

If $n$ is even, that is $n = 2m$, then

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right) = b^{2m}\left(z^{4m}-2z^{2m}+1\right) = b^{2m}\left(\left(-\frac{a}{b}\right)^{2m} - 2\left(-\frac{a}{b}\right)^{m}+1\right)$$
$$= a^{2m}-2a^m\left(-b\right)^m + b^{2m} = \left(a^m-\left(-b\right)^m\right)^2.$$

If $n$ is odd then

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right) = b^n\left(-z^{2n}+1\right) = b^n\left(-\left(-\frac{a}{b}\right)^n+1\right) = a^n+b^n.$$

Thus,

$$\prod_{k=1}^{n-1}\left(a+b\varepsilon_k^2\right) = \begin{cases} \left(a^{\frac{n}{2}}-(-b)^{\frac{n}{2}}\right)^2 \text{ if } n \text{ is even} \\ a^n+b^n \text{ if } n \text{ is odd}\end{cases}.$$

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