Olympiad problems

O145. Find all positive integers n for which

$$\left(1^4+\frac{1}{4}\right)\left(2^4+\frac{1}{4}\right)\cdots\left(n^4+\frac{1}{4}\right)$$

is the square of a rational number.

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Let $P=\prod_{k=1}^{n}\left(k^4+\frac{1}{4}\right)$ and let $a_k=k^2-k+\frac{1}{2}, k=1,2,\ldots,n$. Since $a_{k+1}=k^2+k+\frac{1}{2}$ and $k^4+\frac{1}{4}=\left(k^2+\frac{1}{4}\right)-k^2=a_ka_{k+1}, k=1,2,\ldots,n$ then

$$P=a_1a_{n+1}Q^2=\frac{1}{4}\cdot\left(2n^2+2n+1\right)Q^2,$$

where $Q=\prod_{k=2}^{n}a_k$. Therefore, P is the square of a rational number if and only if $2n^2+2n+1$ is the square of a positive integer, i.e. if and only if $2n^2+2n+1=m^2$ for some positive integer m. Therefore $2m^2-(2n+1)^2=1$ and then our problem is finding the solutions to the equation

$$x^2-2y^2=-1$$

in positive integers. Let

$$\mathbb{Z}\left(\sqrt{2}\right)=\left\{x+y\sqrt{2} \vdots x,y\in\mathbb{Z}\right\},\ \mathbb{N}\left(\sqrt{2}\right):=\left\{x+y\sqrt{2} \vdots x,y\in\mathbb{N}\right\}$$

and let $s=3+2\sqrt{2}$ and for any $z=x+y\sqrt{2}\in\mathbb{Z}\left(\sqrt{2}\right)$ denote $\overline{z}=x-y\sqrt{2}$. In this notation equation $x^2-2y^2=-1,\ \ x,y\in\mathbb{N}$ becomes $z\overline{z}=-1, z\in\mathbb{N}\left(\sqrt{2}\right)$. Denote the set of all such solutions by Sol, i.e.

$$Sol=\left\{z \vdots z\in\mathbb{N}\left(\sqrt{2}\right) \text{ and } z\overline{z}--1\right\}.$$

Note that for $z_0=1+\sqrt{2}\in\mathbb{N}\left(\sqrt{2}\right)$ we have $z_0\overline{z}_0=-1$ and since $s\overline{s}=1$ then for $z_k=s^kz_0$ we also have $z_k\overline{z}_k=-1,\ \ k\in\mathbb{N}$. Also it is clear that z_0 is smallest element in Sol. Note that if $z \in Sol$ (that is $z\overline{z} = -1, z \in \mathbb{N}\left(\sqrt{2}\right)$) and $z \neq z_0$ then $\overline{s}z \in Sol$. Indeed, $\overline{s}z \cdot s\overline{z} = \overline{s}s \cdot z\overline{z} = 1 \cdot (-1) = -1$. It remains to prove $\overline{s}z \in \mathbb{N}\left(\sqrt{2}\right)$. Let $z = x + y\sqrt{2}$ then $x^2 = 2y^2 - 1$ and

$$\overline{s}z = \left(3 - 2\sqrt{2}\right)\left(x + y\sqrt{2}\right) = 3x - 4y + (-2x + 3y)\sqrt{2}.$$

Since $z \neq z_0 \implies z > z_0 \implies x, y \geq 2$, and moreover, $x, y \geq 3$ because x is odd and $2y^2 - 1$ isn't square of integer for $y = 2$, we have

$$3x \geq 4y \iff 9x^2 \geq 16y^2 \iff 9\left(2y^2 - 1\right) \geq 16y^2 \iff 2y^2 \geq 9 \impliedby y \geq 3$$

and

$$3y \geq 2x \iff 9y^2 \geq 4x^2 \iff 9y^2 \geq 4\left(2y^2 - 1\right) \impliedby y^2 + 4 \geq 0.$$

We will prove that $Sol = \left\{z_k \colon k \in \mathbb{N} \cup \{0\}\right\}$. Suppose that exist $z \in Sol$ which not belong to the sequence $z_0 < z_1 < z_2 < \cdots < z_k <, \ldots$. Since $(z_k)_{k \geq 0}$ is unbounded from above then there is k such that $z_k < z < z_{k+1}$. Since $z_0 < \overline{s}^k z < z_1 = sz_0 = 7 + 5\sqrt{2}$ then $\overline{s}^{k+1} z < z_0$ and $\overline{s}^{k+1} z \in Sol$. That is the contradiction because z_0 is smallest element in Sol. Thus,

$$Sol = \left\{z_k \colon k \in \mathbb{N} \cup \{0\}\right\}.$$

Let $z_k = x_k + y_k\sqrt{2}, k \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} z_{k+1} = sz_k \iff & \begin{cases} x_{k+1} = 3x_k + 4y_k \\ y_{k+1} = 2x_k + 3y_k \end{cases} \\ \implies & x_{k+2} - 3x_{k+1} = 2x_k + 3\left(x_{k+1} - 3x_k\right) \\ \iff & x_{k+2} - 6x_{k+1} + 7x_k = 0, x_0 = 1, x_1 = 7. \end{aligned}$$

Since $x = 2n + 1$ then all natural n for which $2n^2 + 2n + 1$ is a square of integer should be elements of set $\left\{n_k \colon n_k = \dfrac{x_k - 1}{2},\ k = 1, 2, \ldots\right\}$. By substitution $x_k = 2n_k + 1$ in $x_{k+2} - 6x_{k+1} + 7x_k = 0$ we obtain

$$2n_{k+2} + 1 - 12n_{k+1} - 6 + 14n_k + 7 = 0 \iff n_{k+2} = 6n_{k+1} - 7n_k - 1$$

Thus, all solutions of problem are the terms of the sequence $(n_k)_{k \geq 1}$ defined recursively by

$$n_{k+2} = 6n_{k+1} - 7n_k - 1, n_0 = 0, n_1 = 3.$$

In particular, $n_1 = 3, n_2 = 17, n_3 = 80, n_4 = 360, \ldots$