O101. Let $a_0, a_1, \ldots, a_6$ be real numbers greater than -1. Prove that

$$\frac{a_0^2+1}{\sqrt{a_1^5+a_1^4+1}}+\frac{a_1^2+1}{\sqrt{a_2^5+a_2^4+1}}+\cdots+\frac{a_6^2+1}{\sqrt{a_0^5+a_0^4+1}}\geq 5$$

whenever

$$\frac{a_0^3+1}{\sqrt{a_1^5+a_1^4+1}}+\frac{a_1^3+1}{\sqrt{a_2^5+a_2^4+1}}+\cdots+\frac{a_6^3+1}{\sqrt{a_0^5+a_0^4+1}}\leq 9.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

It suffices to prove that

$$\frac{a_0^3+a_0^2+2}{\sqrt{a_1^5+a_1^4+1}}+\frac{a_1^3+a_1^2+2}{\sqrt{a_2^5+a_2^4+1}}+\ldots+\frac{a_n^3+a_n^2+2}{\sqrt{a_0^5+a_0^4+1}}\geq 14$$

We observe that $(a_{n+1} \doteq a_0)$

$$\sum_{k=0}^{6}\frac{a_k^3+a_k^2+2}{\sqrt{a_{k+1}^5+a_{k+1}^4+1}}=\sum_{k=0}^{6}\frac{a_k^3+a_k^2+2}{\sqrt{a_{k+1}^2+a_{k+1}+1}\sqrt{a_{k+1}^3-a_{k+1}+1}}$$

which we rewrite as

$$\sum_{k=0}^{6}\frac{a_k^3-a_k+1}{\sqrt{a_{k+1}^2+a_{k+1}+1}\sqrt{a_{k+1}^3-a_{k+1}+1}}+\sum_{k=0}^{6}\frac{a_k^2+a_k+1}{\sqrt{a_{k+1}^2+a_{k+1}+1}\sqrt{a_{k+1}^3-a_{k+1}+1}}$$

namely

$$\sum_{k=0}^{6}\frac{\sqrt{a_k^3-a_k+1}}{\sqrt{a_{k+1}^2+a_{k+1}+1}}+\sum_{k=0}^{6}\frac{\sqrt{a_k^2+a_k+1}}{\sqrt{a_{k+1}^3-a_{k+1}+1}}\geq 2\cdot 7=14$$

the last inequality allowed by the AGM and we are done.

Second solution by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

We have the identity

$$a^5+a^4+1=(a^2+a+1)(a^3-a+1)$$

Assume by contradiction that

$$\frac{a_0^3+1}{a_1^5+a_1^4+1}+\frac{a_1^3+1}{a_2^5+a_2^4+1}+\cdots+\frac{a_6^3+1}{a_0^5+a_0^4+1}\leq 9$$

Let $x_i := a_i^3 + a_i^2 + 2,\ y_i := \dfrac{1}{\sqrt{a_i^5 + a_i^4 + 1}}, i = 0, 1, ..., 6.$

Since $sign\left((x_i - x_j)(y_j - y_i)\right) = sign\left((x_i - x_j)\left(y_j^2 - y_i^2\right)\right) = sign\left((a_i - a_j)^2\right)$ then, applying Rearrangement Inequality and inequality (1) obtain

$A_2 + A_3 = x_0y_1 + x_1y_2 + ... + x_6y_0 \geq \sum_{i=0}^{6} x_iy_i = \sum_{i=0}^{6} \dfrac{a_i^3 + a_i^2 + 2}{\sqrt{a_i^5 + a_i^4 + 1}} \geq 7 \cdot 2 =$ 14.

Since $A_2 + A_3 \geq 14$ and $A_3 \leq 9$ then $A_2 \geq 14 - A_3 \geq 14 - 9 = 5$.

Fourth solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Clearly $(x^3 - x + 1)(x^2 + x + 1) = x^5 + x^4 + 1$, or using the AM-GM inequality, $x^3 + x^2 + 2 \geq 2\sqrt{x^5 + x^4 + 1}$ for all non-negative reals x. This is true in particular for all the a_i, hence

$$\left(\frac{a_0^2 + 1}{\sqrt{a_1^5 + a_1^4 + 1}} + \cdots + \frac{a_6^2 + 1}{\sqrt{a_0^5 + a_0^4 + 1}}\right) + \left(\frac{a_0^3 + 1}{\sqrt{a_1^5 + a_1^4 + 1}} + \cdots + \frac{a_6^3 + 1}{\sqrt{a_0^5 + a_0^4 + 1}}\right)$$

$$\geq 2\left(\frac{\sqrt{a_0^5 + a_0^4 + 1}}{\sqrt{a_1^5 + a_1^4 + 1}} + \cdots + \frac{\sqrt{a_6^5 + a_6^4 + 1}}{\sqrt{a_0^5 + a_0^4 + 1}}\right) \geq 14,$$

since the bracket in the middle term is the sum of 7 elements of product 1, and the AM-GM inequality has been used again. Therefore, the sum of both elements is not smaller than 14, and if one of them does not exceed 9, then the other one is at least 5. The result follows.

Note that we may use the AM-GM inequality since, for all $x \geq -1$, it holds $x^3 - x + 1 > 0$ and $x^2 + x + 1 > 0$. In fact, the proposed result would be true, not only for $a_i \geq -1$, but for all a_i larger than the negative real root of $x^3 - x + 1$. Note also that 5 and 9 may be exchanged by any pair of non-negative real numbers such that their sum is 14.