J99. In a triangle ABC, let ϕ_a, ϕ_b, ϕ_c be the angles between medians and altitudes emerging from the same vertex. Prove that one of the numbers $\tan \phi_a, \tan \phi_b, \tan \phi_c$ is the sum of the other two.

Proposed by Oleh Faynshteyn, Leipzig, Germany

First solution by Arkady Alt, San Jose, California, USA

If we define ϕ_a, ϕ_b, ϕ_c as oriented angles between medians and altitudes (let it be counterclockwise orientation) then statement of problems becomes Prove that $\tan \phi_a + \tan \phi_b + \tan \phi_c = 0$. Since $a = b \cos C + c \cos B$ and $\tan \phi_a = \dfrac{\frac{a}{2} - c \cos B}{c \sin B}$ then, applying the Sine Theorem we obtain

$$\begin{aligned}\tan \phi_a &= \frac{b \cos C - c \cos B}{c \sin B} \\ &= \frac{2R \sin B \cos C - 2R \sin C \cos B}{2R \sin C \sin B} \\ &= \frac{\sin B \cos C - \sin C \cos B}{\sin C \sin B} \\ &= \cot C - \cot B\end{aligned}$$

therefore

$$\sum_{cyc} \tan \phi_a = \sum_{cyc} (\cot C - \cot B) = 0.$$

Second solution by Magkos Athanasios, Kozani, Greece

Let AM be the median and AD the altitude emerging from vertex A. It is obvious that

$$\tan \phi_a = \frac{MD}{AD}.$$

Recalling that $|b^2 - c^2| = 2aMD$, we obtain

$$\tan \phi_a = \frac{|b^2 - c^2|}{2ah_a} = \frac{|b^2 - c^2|}{4F},$$

where F is the area of triangle ABC. Similarly, we have

$$\tan \phi_b = \frac{|c^2 - a^2|}{4F}, \quad \tan \phi_c = \frac{|a^2 - b^2|}{4F}.$$

Without loss of generality assume that $a \geq b \geq c$. We find then that $\tan \phi_b = \tan \phi_a + \tan \phi_c$.