J90. For a fixed positive integer n let $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that

$$(a_1 - a_0)\cdots(a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

First solution by Arkady Alt, San Jose, California, USA

Let $a := 2^{2^{-n}}$ then $a^{2^k} = \left(2^{2^{-n}}\right)^{2^k} = 2^{2^{k-n}}$ and, therefore,

$a_k - a_{k-1} = 2^{2^{k-n}} + k - 2^{2^{k-n-1}} - (k-1) = 2^{2^{k-n}} - 2^{2^{k-n-1}} + 1 = a^{2^k} - a^{2^{k-1}} + 1 = \dfrac{a^{2^{k+1}} + a^{2^k} + 1}{a^{2^k} + a^{2^{k-1}} + 1}, k = 0, 1, 2, ..., n$.

Thus, $\prod_{k=1}^{n}(a_k - a_{k-1}) = \prod_{k=1}^{n} \dfrac{a^{2^{k+1}} + a^{2^k} + 1}{a^{2^k} + a^{2^{k-1}} + 1} = \dfrac{a^{2^{n+1}} + a^{2^n} + 1}{a^{2^1} + a^{2^0} + 1} = \dfrac{2^{2^{n+1-n}} + 2^{2^{n-n}} + 1}{2^{2^{1-n}} + 1 + 2^{2^{0-n}} + 0} = \dfrac{7}{a_1 + a_0}.$

Second solution by Brian Bradie, VA, USA

We start with $a_n = n + 2$ and $a_{n-1} = n - 1 + \sqrt{2}$. Therefore

$$a_n - a_{n-1} = 3 - \sqrt{2} = \frac{7}{3 + \sqrt{2}} = \frac{7}{a_n + a_{n-1} + 2(n-1)}.$$

Now,

$$\begin{aligned}(a_k - a_{k-1})(a_k + a_{k-1} - 2(k-1)) &= \left(2^{2^{k-n}} + 1 - 2^{2^{k-1-n}}\right)\left(2^{2^{k-n}} + 1 + 2^{2^{k-1-n}}\right) \\ &= 2^{2^{k+1-n}} + 2 \cdot 2^{2^{k-n}} + 1 - 2^{2^{k-n}} \\ &= 2^{2^{k+1-n}} + k + 1 + 2^{2^{k-n}} + k - 2k \\ &= a_{k+1} + a_k - 2k,\end{aligned}$$

so

$$a_k - a_{k-1} = \frac{a_{k+1} + a_k - 2k}{a_k + a_{k-1} - 2(k-1)}.$$

Thus,

$$\begin{aligned}(a_{n-1} - a_{n-2})(a_n - a_{n-1}) &= \frac{7}{a_n + a_{n-1} + 2(n-1)} \cdot \frac{a_n + a_{n-1} + 2(n-1)}{a_{n-1} + a_{n-2} + 2(n-2)} \\ &= \frac{7}{a_{n-1} + a_{n-2} + 2(n-2)}\end{aligned}$$