J74. A triangle has altitudes h_a, h_b, h_c and inradius r. Prove that

$$\frac{3}{5} \le \frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} < \frac{3}{2}.$$

Proposed by Oleh Faynshteyn, Leipzig, Germany

First solution by Arkady Alt, San Jose, California, USA

Let s be the semiperimeter in our triangle. Since $2rs = ah_a$ then $\frac{2r}{h_a} = \frac{a}{s}$ and

$$\sum_{cyc} \frac{h_a - 2r}{h_a + 2r} = \sum_{cyc} \frac{s-a}{s+a} = 2s \sum_{cyc} \frac{1}{s+a} - 3.$$

Thus

$$\frac{3}{5} \le \sum_{cyc} \frac{h_a - 2r}{h_a + 2r} \iff 9 \le 5s \sum_{cyc} \frac{1}{s+a} = \sum_{cyc} (s+a) \cdot \sum_{cyc} \frac{1}{s+a},$$

where the latter inequality is an application of the Cauchy Inequality to triples

$$\left(\sqrt{s+a}, \sqrt{s+b}, \sqrt{s+c}\right), \left(\frac{1}{\sqrt{s+a}}, \frac{1}{\sqrt{s+b}}, \frac{1}{\sqrt{s+c}}\right).$$

Instead of proving that $\sum_{cyc} \dfrac{h_a - 2r}{h_a + 2r} < \dfrac{3}{2}$, we prove that $\sum_{cyc} \dfrac{h_a - 2r}{h_a + 2r} < 1$.

Let $x = s - a, y = s - b, z = s - c$, $x, y, z > 0$. Due to homogeneity we can assume that $s = x + y + z = 1$, then $a = 1 - x, b = 1 - y, c = 1 - z$. Thus,

$$\begin{aligned} \sum_{cyc} \frac{h_a - 2r}{h_a + 2r} < 1 &\iff \sum_{cyc} \frac{s-a}{s+a} < 1 \\ &\iff s \sum_{cyc} \frac{1}{s+a} < 2 \\ &\iff \sum_{cyc} \frac{1}{2-x} < 2 \\ &\iff \sum_{cyc} (2-y)(2-z) < 2(2-x)(2-y)(2-z). \end{aligned}$$

The last expression is equivalent to

$$\sum_{cyc} (4 - 2(y+z) + yz) < 2(8 - 4(x+y+z) + 2(xy+yz+zx) - xyz),$$

$$8 + xy + yz + zx < 8 + 4(xy + yz + zx) - 2xyz \iff 2xyz < 3(xy + yz + zx),$$

and the latter inequality holds since

$$3(xy+yz+zx)=3\left(x+y+z\right)\left(xy+yz+zx\right)\geq 27xyz>2xyz.$$

Second solution by G.R.A.20 Math Problems Group, Roma, Italy

We know that

$$2A=h_a a=h_b b=h_c c=sr$$

where s is the semiperimeter and A is the area of the triangle. By replacing h_a, h_b, h_c, r in the original equation we obtain

$$\frac{3}{5}\leq\frac{s-a}{s+a}+\frac{s-b}{s+b}+\frac{s-c}{s+c}<\frac{3}{2}$$

that is

$$\frac{3}{5}\leq\frac{x}{x+2y+2z}+\frac{y}{y+2z+2z}+\frac{z}{z+2x+2y}<\frac{3}{2}$$

where $x=s-a$, $y=s-b$, $z=s-c$ are non-negative (and not all zero otherwise the triangle becomes a point).

The inequality on the left is equivalent to

$$2\sum_{\text{sym}}x^2y+2\sum_{\text{sym}}xyz\leq 4\sum_{\text{sym}}x^3$$

which holds by Muirhead's inequality (the equality holds when $x=y=z$, that is when the triangle is equilateral).

The inequality on the right is equivalent to

$$0<4\sum_{\text{sym}}x^3+52\sum_{\text{sym}}x^2y+19\sum_{\text{sym}}xyz$$

which holds because x, y and z are non-negative and not all zero.

Third solution by Salem Malikic, Sarajevo, Bosnia and Herzegovina

Let P denote the area of a triangle. Then

$$\frac{h_a-2r}{h_a+2r}=1-\frac{4r}{h_a+2r}=1-\frac{\frac{8P}{a+b+c}}{\frac{2P}{a}+\frac{4P}{a+b+c}}=1-\frac{\frac{4}{a+b+c}}{\frac{1}{a}+\frac{2}{a+b+c}}=1-\frac{4a}{3a+b+c}.$$

Let us prove the right side of the inequality first.

$$\sum\frac{h_a-2r}{h_a+2r}=3-\sum\frac{4a}{3a+b+c}<\frac{3}{2},$$