J66. Let $a_0 = a_1 = 1$ and $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n \geq 1$. Prove that $a_{n^2+1} = a_{n+1}a_n$ for all $n \geq 0$.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by O.O.Ibrogimov, Samarqand State University, Uzbekistan

Because $a_{n+1} + a_{n-1} = 2a_n + 2$, we have

$$\begin{aligned} a_2 + a_0 &= 2a_1 + 2 \\ a_3 + a_1 &= 2a_2 + 2 \\ &\dots \\ a_{m-1} + a_{m+1} &= 2a_m + 2 \end{aligned}$$

Summing up we get

$$a_0 + a_1 + 2(a_2 + \ldots + a_{m-1}) + a_m + a_{m+1} = 2(a_1 + a_2 + \ldots + a_m) + 2m,$$

yielding

$$a_{m+1} = a_m + m.$$

From here it is not difficult to find that $a_m = m^2 - m + 1$. Then

$$a_{n^2+1} = (n^2+1)^2 - (n^2+1) + 1 = ((n+1)^2 - (n+1) + 1)(n^2 - n + 1) = a_{n+1}a_n.$$

Second solution by Arkady Alt, San Jose, California, USA

Observe that $a_{n+1} - a_n - 2n = a_n - a_{n-1} - 2(n-1)$, for $n \geq 1$. Therefore, $a_{n+1} - a_n - 2n = c$, where c is some constant. Because $a_{n+1} - a_n - 2n = c$ we can conclude $a_n = (n-1)\,n + cn + b$, for $n \geq 0$. Initial conditions $a_0 = a_1 = 1$ give us $c = 0$ and $b = 1$, i.e. $a_n = n^2 - n + 1$, for $n \geq 0$. Hence

$$a_{n^2+1} = \left(n^2+1\right)^2 - \left(n^2+1\right) + 1 = \left(n^2 - n + 1\right)\left(n^2 + n + 1\right) = a_n a_{n+1}.$$

Third solution by Brian Bradie, Christopher Newport University, USA

The characteristic equation associated with the difference equation $a_{n+1} = 2a_n - a_{n-1}$ has a double root of 1; therefore, the complementary solution associated with the difference equation $a_{n+1} = 2a_n - a_{n-1} + 2$ is

$$c_1 + c_2 n$$