J45. Let a and b be real numbers. Find all pairs (x, y) of real numbers solutions to the system

$$\begin{cases} x+y &= \sqrt[3]{a+b} \\ x^4-y^4 &= ax-by \end{cases}$$

Proposed by Dr. Titu Andreescu, University of Texas at Dallas.

First solution by Arkady Alt, San Jose, California, USA

First, we will solve the original system with respect to a and b.

$$\begin{cases} a+b=(x+y)^3 \\ ax-by=x^4-y^4 \end{cases} \iff \begin{cases} a(x+y)=y(x+y)^3+x^4-y^4 \\ b(x+y)=x(x+y)^3-x^4+y^4 \end{cases} \iff$$

$$\begin{cases} a(x+y)=x(x+y)\left(x^2+3y^2\right) \\ b(x+y)=y(x+y)\left(y^2+3x^2\right) \end{cases}$$

$x+y$ can be equal to zero only if $a=-b$ and in this case the original system has infinitely many solutions $(t,-t), t\in\mathbb{R}$. Supposing that $a+b\neq 0$, we have $x+y\neq 0$. Thus, we obtain

$$\begin{cases} a=x\left(x^2+3y^2\right) \\ b=y\left(y^2+3x^2\right) \end{cases} \iff \begin{cases} a+b=(x+y)^3 \\ a-b=(x-y)^3 \end{cases} \iff \begin{cases} x+y=\sqrt[3]{a+b} \\ x-y=\sqrt[3]{a-b} \end{cases}$$

$$\iff \begin{cases} x=\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2} \\ y=\frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2} \end{cases}.$$

Second solution by Daniel Campos Salas, Costa Rica

Using that $x^4-y^4=(x-y)((x+y)^3-2xy(x+y))$, we have

$$(x-y)(a+b-2xy(x+y))=ax-by.$$

This implies that

$$bx-ay=2xy(x+y)(x-y)=2xy(x^2-y^2).$$

Then,

$$\begin{aligned}(a-b)(x+y) &= (ax-by)-(bx-ay)=(x^4-y^4)-2xy(x^2-y^2) \\ &= (x^2-y^2)(x-y)^2=(x+y)(x-y)^3.\end{aligned}$$

If $a+b\neq 0$ then $x+y\neq 0$, so