

Junior problems

J409. Solve the equation

$$\log(1 - 2^x + 5^x - 20^x + 50^x) = 2x.$$

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Let $u := 2^x, v := 5^x$. Then $u, v > 0$ and

$$\log(1 - 2^x + 5^x - 20^x + 50^x) = 2x \iff 1 - 2^x + 5^x - 20^x + 50^x = 10^{2x} \iff$$

$$1 - u + v - u^2v + uv^2 = u^2v^2 \iff 1 - u + v - u^2v + uv^2 - u^2v^2 = 0 \iff$$

$$(1 - uv)(1 + uv) - (u - v)(1 + uv) = 0 \iff (1 + uv)(1 - uv - u + v) = 0 \iff$$

$$(1 + uv)(1 + v)(1 - u) = 0 \iff u = 1$$

because $1 + uv, 1 + v > 0$.

Hence $2^x = 1 \iff x = 0$. Thus, $x = 0$ is only solution.

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