Junior problems

J301. Let a and b be nonzero real numbers such that $ab \geq \frac{1}{a} + \frac{1}{b} + 3$. Prove that

$$ab \geq \left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}}\right)^3.$$

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Let $x := \sqrt[3]{ab}, y := -\dfrac{1}{\sqrt[3]{a}}, z := -\dfrac{1}{\sqrt[3]{b}}$

Then $xyz = 1$ and $ab \geq \dfrac{1}{a} + \dfrac{1}{b} + 3 \iff x^3 + y^3 + z^3 \geq 3.$

Since

$$(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right) = x^3+y^3+z^3-3xyz = x^3+y^3+z^3-3 \geq 0$$

and

$$x^2+y^2+z^2 \geq xy+yz+zx$$

$$x+y+z \geq 0 \iff x \geq -(y+z) \iff x^3 \geq (-y-z)^3 \iff ab \geq \left(\frac{1}{\sqrt[3]{a}} + \frac{1}{\sqrt[3]{b}}\right)^3.$$

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