J297. Let a, b, c be digits in base $x \geq 4$. Prove that

$$\frac{\overline{ab}}{\overline{ba}} + \frac{\overline{bc}}{\overline{cb}} + \frac{\overline{ca}}{\overline{ac}} \geq 3,$$

where all numbers are written in base x.

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Note that $\frac{\overline{ab}}{\overline{ba}} + \frac{\overline{bc}}{\overline{cb}} + \frac{\overline{ca}}{\overline{ac}} \geq 3$ can be written as $\frac{ax+b}{bx+a} + \frac{bx+c}{cx+b} + \frac{cx+a}{ax+c} \geq 3$.

Since by Cauchy Inequality

$$\sum_{cyc} \frac{ax+b}{bx+a} = \sum_{cyc} \frac{(ax+b)^2}{(ax+b)(bx+a)} \geq \frac{\left(\sum_{cyc}(ax+b)\right)^2}{\sum_{cyc}(ax+b)(bx+a)} =$$

$$\frac{(a+b+c)^2(x+1)^2}{\sum_{cyc}\left(abx^2 + (a^2+b^2)x + ab\right)} = \frac{(a+b+c)^2(x+1)^2}{(ab+bc+ca)x^2 + 2(a^2+b^2+c^2)x + ab+bc+ca}$$

it suffice to prove inequality

$$(a+b+c)^2(x+1)^2 \geq 3\left((ab+bc+ca)x^2 + 2\left(a^2+b^2+c^2\right)x + ab+bc+ca\right).$$

We have

$$(a+b+c)^2(x+1)^2 - 3\left((ab+bc+ca)x^2 + 2\left(a^2+b^2+c^2\right)x + ab+bc+ca\right) =$$

$\left(a^2+b^2+c^2-ab-bc-ca\right)\left(x^2-4x+1\right) \geq 0$ because $a^2+b^2+c^2 \geq ab+bc+ca$ and

$$x^2-4x+1 = x(x-4)+1 \geq 1.$$

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