Junior problems

J277. Is there an integer n such that $4^{5^n} + 5^{4^n}$ is a prime?

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The answer is – No because $4^{5^n} + 5^{4^n}$ can be represented in the form:

$$a^4 + 4b^4 = \left(a^2 - 2ab + 2b^2\right)\left(a^2 + 2ab + 2b^2\right).$$

Indeed, since $5^n - 1 = (5-1)\left(5^{n-1} + 5^{n-2} + ... + 5 + 1\right) = 4\left(5^{n-1} + 5^{n-2} + ... + 5 + 1\right)$ then $5^{4^n} + 4^{5^n} = \left(5^{4^{n-1}}\right)^4 + 4 \cdot \left(4^{5^{n-1}+5^{n-2}+...+5+1}\right)^4 = a^4 + 4b^4$, where

$$a := 5^{4^{n-1}} \text{and } b := 4^{5^{n-1}+5^{n-2}+...+5+1}$$

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