J243. Let a, b, c be real numbers such that

$$\left(-\frac{a}{2}+\frac{b}{3}+\frac{c}{6}\right)^3+\left(\frac{a}{3}+\frac{b}{6}-\frac{c}{2}\right)^3+\left(\frac{a}{6}-\frac{b}{2}+\frac{c}{3}\right)^3=\frac{1}{8}.$$

Prove that

$$(a-3b+2c)(2a+b-3c)(-3a+2b+c)=9.$$

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Notice that

$$\left(-\frac{a}{2}+\frac{b}{3}+\frac{c}{6}\right)^3+\left(\frac{a}{3}+\frac{b}{6}-\frac{c}{2}\right)^3+\left(\frac{a}{6}-\frac{b}{2}+\frac{c}{3}\right)^3=\frac{1}{8} \text{ iff } x^3+y^3+z^3=27,$$

where

$$x:=(-3a+2b+c)^3,\ y:=(2a+b-3c)^3,\ z:=(a-3b+2c)^3.$$

Since for any reals x, y, z, we know that

$$x^3+y^3+z^3-3xyz=(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right),$$

it follows in our case, where we know $x+y+z=0$, that $x^3+y^3+z^3-3xyz=0$, i.e. $xyz=9$. This is precisely what we want.

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