J233. Let $A_1A_2A_3A_4A_5$ be a regular pentagon and let $B_1B_2B_3B_4B_5$ be the pentagon formed by its diagonals. Prove that

$$\frac{K_{B_1B_2B_3B_4B_5}}{K_{A_1A_2A_3A_4A_5}} = \frac{7-3\sqrt{5}}{2}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Arkady Alt, San Jose, California, USA

First, recall that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ and $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$. Let a and b be the common value for the sidelengths of pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$. We then have that

$$\frac{K_{B_1B_2B_3B_4B_5}}{K_{A_1A_2A_3A_4A_5}} = \frac{b^2}{a^2}.$$

Now, we know that $A_2A_5 = 2a\cos 36^\circ$ and $x = \dfrac{a}{2\cos 36^\circ}$, thus

$$\begin{aligned} b &= A_2A_5 - 2x = 2a\cos 36^\circ - \frac{a}{\cos 36^\circ} = a\left(2\cos 36^\circ - \frac{1}{\cos 36^\circ}\right) \\ &= \frac{a\left(2\cos^2 36^\circ - 1\right)}{\cos 36^\circ} = \frac{a\cos 72^\circ}{\cos 36^\circ} = \frac{a\sin 18^\circ}{\cos 36^\circ} = a\cdot\frac{\sqrt{5}-1}{\sqrt{5}+1} \\ &= a\cdot\frac{3-\sqrt{5}}{2}, \end{aligned}$$

and, therefore,

$$\frac{b^2}{a^2} = \left(\frac{3-\sqrt{5}}{2}\right)^2 = \frac{7-3\sqrt{5}}{2}.$$

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