Junior problems

J223. Let a and b be real numbers such that $\sin^3 a - \frac{4}{3}\cos^3 a \leq b - \frac{1}{4}$. Prove that

$$\frac{3}{4}\sin a - \cos a \leq b + \frac{1}{6}.$$

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Since

$$\sin^3 a - \frac{4}{3}\cos^3 a \leq b - \frac{1}{4} \iff \sin^3 a - \frac{4}{3}\cos^3 a + \frac{5}{12} \leq b + \frac{1}{6},$$

it suffices to prove that

$$\frac{3}{4}\sin a - \cos a \leq \sin^3 a - \frac{4}{3}\cos^3 a + \frac{5}{12} \iff 4\sin a - 12\cos a - 12\sin^3 a + 16\cos^3 a \leq 5.$$

But

$$9\sin a - 12\cos a - 12\sin^3 a + 16\cos^3 a = 3\left(3\sin a - 4\sin^3 a\right) + 4\left(4\cos^3 a - 3\cos a\right)$$

and this equals

$$3\sin 3a + 4\cos 3a = 5\left(\sin 3a \cdot \frac{3}{5} + \cos 3a \cdot \frac{4}{5}\right) = 5\sin(3a + \varphi) \leq 5,$$

where

$$\cos\varphi = \frac{3}{5}, \sin\varphi = \frac{4}{5}.$$

(either that or simply by Cauchy Inequality write: $3\sin 3a + 4\cos 3a \leq \sqrt{3^2 + 4^2} \cdot \sqrt{\sin^2 a + \cos^2 a} = 5$).

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