J197. Let x, y, z be positive real numbers. Prove that

$$\sqrt{2\left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)} \geq x\sqrt{\frac{1}{y}+\frac{1}{z}}+y\sqrt{\frac{1}{z}+\frac{1}{x}}+z\sqrt{\frac{1}{x}+\frac{1}{y}}.$$

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Note that by the Cauchy-Schwarz Inequality we have that

$$\sum_{cyc} x\sqrt{\frac{1}{y}+\frac{1}{z}} \leq \sqrt{\sum_{cyc} x^2 \sum_{cyc}\left(\frac{1}{y}+\frac{1}{z}\right)} = \sqrt{2\sum_{cyc} x^2 \sum_{cyc}\frac{1}{x}},$$

so it would suffice to prove that

$$\left(x^2y^2+y^2z^2+z^2x^2\right)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) \geq \left(x^2+y^2+z^2\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Now, let $a := \frac{1}{x}, b := \frac{1}{y}, c := \frac{1}{z}$; then the inequality to be proven can be rewritten as

$$\left(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2}\right)\left(a^3+b^3+c^3\right) \geq \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c),$$

which is equivalent with

$$\left(a^2+b^2+c^2\right)\left(a^3+b^3+c^3\right) \geq \left(a^2b^2+b^2c^2+c^2a^2\right)(a+b+c),$$

i.e.

$$\sum_{cyc} a^5 + \sum_{cyc} a^3\left(b^2+c^2\right) \geq \sum_{cyc} a^3\left(b^2+c^2\right) + \sum_{cyc} ab^2c^2,$$

which turns out to be just the immediate

$$\sum_{cyc} a^5 \geq \sum_{cyc} ab^2c^2 = abc\,(ab+bc+ca),$$

which can be seen for example as a consequence of the AM-GM Inequality.

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