J197. Proposed by Vazgen Mikayelyan, Yerevan, Armenia.

Let x,y,z be positive real numbers. Prove that

$$\sqrt{2(x^2y^2+y^2z^2+z^2x^2)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)} \ge x\sqrt{\frac{1}{y}+\frac{1}{z}}+y\sqrt{\frac{1}{z}+\frac{1}{x}}+z\sqrt{\frac{1}{x}+\frac{1}{y}}.$$

Solution by Arkady Alt , San Jose ,California, USA.

Since by Cauchy Inequality

$$\sum_{cyc} x\sqrt{\frac{1}{y}+\frac{1}{z}} \le \sqrt{\sum_{cyc} x^2 \sum_{cyc}\left(\frac{1}{y}+\frac{1}{z}\right)} = \sqrt{2\sum_{cyc} x^2 \sum_{cyc}\frac{1}{x}}$$

then suffice to prove inequality

(1) $(x^2y^2+y^2z^2+z^2x^2)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) \ge (x^2+y^2+z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$

Let $a := \frac{1}{x}, b := \frac{1}{y}, c := \frac{1}{z}$ then (**1**) becomes

$$\left(\frac{1}{a^2b^2}+\frac{1}{b^2c^2}+\frac{1}{c^2a^2}\right)(a^3+b^3+c^3) \ge \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c) \Leftrightarrow$$

$$(a^2+b^2+c^2)(a^3+b^3+c^3) \ge (a^2b^2+b^2c^2+c^2a^2)(a+b+c) \Leftrightarrow$$

$$\sum_{cyc} a^5 + \sum_{cyc}. a^3(b^2+c^2) \ge \sum_{cyc}. a^3(b^2+c^2) + \sum_{cyc} ab^2c^2 \Leftrightarrow \sum_{cyc} a^5 \ge \sum_{cyc} ab^2c^2 \Leftrightarrow$$

(2) $a^5+b^5+c^5 \ge abc(ab+bc+ca).$

Since by AM-GM Inequality $2a^5+2b^5+c^5 \ge 5\sqrt[5]{a^{10}b^{10}c^5} = 5a^2b^2c$

then $\sum_{cyc}(2a^5+2b^5+c^5) \ge \sum_{cyc} 5a^2b^2c \Leftrightarrow a^5+b^5+c^5 \ge abc(ab+bc+ca).$

Or, combining Chebishev's inequality, AM-GM inequality and inequality $a^2+b^2+c^2 \ge ab+bc+ca$ we obtain $a^5+b^5+c^5 \ge \frac{(a^3+b^3+c^3)(a^2+b^2+c^2)}{3} \ge \frac{3abc(ab+bc+ca)}{3} = abc(ab+bc+ca).$