J194. Let a, b, c be the side-lenghts of a triangle with the largest side c. Prove that

$$\frac{ab\,(2c+a+b)}{(a+c)\,(b+c)} \leq \frac{a+b+c}{3}.$$

Proposed by Arkady Alt, San Jose, California, USA

First solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Without loss of generality assume that $c \geq a \geq b$. Consider the function

$$f(x) = \frac{a+b+x}{3} - ab\left(\frac{1}{a+x} + \frac{1}{b+x}\right).$$

We have

$$f(c) - f(a) = (c-a)\left(\frac{1}{3} + \frac{ab}{2a(a+c)} + \frac{1}{(b+a)(b+c)}\right) \geq 0 \tag{1}$$

and

$$f(a) = \frac{2a+b}{3} - ab(\frac{1}{2a} + \frac{1}{a+b}) \geq \frac{2a+b}{3} - \frac{b}{2} - \frac{ab}{4}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{5}{12}(a-b) \geq 0.$$

Note that, the given inequality holds for any positive a, b, c with the largest c.

Second solution by Hoang Quoc Viet, University of Auckland, New Zealand

Assume that $c = \max\{a, b, c\}$, hence

$$a(c-b) + b(c-a) \geq 0. \tag{1}$$

The original inequality can be written as

$$\frac{3abc}{a+b+c} \leq c^2 + a(c-b) + b(c-a).$$

We have

$$\frac{3abc}{a+b+c} \leq c\left(\frac{c^2 + c(a+b)}{a+b+c}\right) = c^2. \tag{2}$$

Combining (1) and (2) the inequality is proved.

Also solved by Arber Selimi, Bedri Pejani - Peje, Kosovo; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Ercole Suppa, Teramo, Italy; Henry Ricardo New York, USA; Mihai Stoenescu, Bischwiller, France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Roberto Bosch Cabrera, Havana, Cuba; Christopher Wiriawan, Jakarta, Indonesia.