J183. Let x, y, z be real numbers. Prove that

$$(x^2+y^2+z^2)^2+xyz(x+y+z) \geq \frac{2}{3}(xy+yz+zx)^2+(x^2y^2+y^2z^2+z^2x^2).$$

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The original inequality will follow from the following sharper inequality

$$\left(x^2+y^2+z^2\right)^2+xyz\left(x+y+z\right)-\left(x^2y^2+y^2z^2+z^2x^2\right) \geq \left(xy+yz+zx\right)^2. \qquad (1)$$

Indeed, for any real u, v, w we have

$$\begin{aligned} u^2+v^2+w^2 \geq uv+vw+wu &\iff (u+v+w)^2 \geq 3\left(uv+vw+wu\right) \\ &\iff (u-v)^2+(v-w)^2+(w-u)^2 \geq 0. \end{aligned}$$

Then

$$\left(x^2+y^2+z^2\right)^2 \geq 3\left(x^2y^2+y^2z^2+z^2x^2\right)$$

and

$$x^2y^2+y^2z^2+z^2x^2 \geq xyz\left(x+y+z\right).$$

Therefore,

$$\begin{aligned} \left(x^2+y^2+z^2\right)^2+xyz\left(x+y+z\right)-\left(x^2y^2+y^2z^2+z^2x^2\right) &\geq 2\left(x^2y^2+y^2z^2+z^2x^2\right)+xyz\left(x+y+z\right) \\ &\geq x^2y^2+y^2z^2+z^2x^2+2xyz\left(x+y+z\right) \\ &= \left(xy+yz+zx\right)^2 \geq \frac{2}{3}\left(xy+yz+zx\right)^2. \end{aligned}$$

Remark. Equality in (1) occurs if and only if $x=y=z$ and in original inequality equality occurs if and only if $x=y=z=0$.

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By expanding the terms $(x^2+y^2+z^2)^2$ and $(xy+yz+zx)^2$ and equating the terms on both sides leads to

$$x^4+y^4+z^4+\frac{1}{3}(x^2y^2+y^2z^2+z^2x^2)-\frac{1}{3}xyz(x+y+z) \geq 0.$$

By the AM-GM inequality we have

$$x^2y^2+y^2z^2+z^2x^2 \geq xyz(x+y+z).$$

Hence it is enough to prove that

$$x^4+y^4+z^4 \geq 0$$

which is obvious. Equality occurs if and onyl if $x=y=z=0$.

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