J178. Find the sequences of integers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that

$$(2+\sqrt{5})^n = a_n + b_n\frac{1+\sqrt{5}}{2}$$

for each $n \geq 0$.

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First solution by Arkady Alt, San Jose, California, USA

Let

$$p_n = \frac{\left(2+\sqrt{5}\right)^n + \left(2-\sqrt{5}\right)^n}{2}$$

and

$$q_n = \frac{\left(2+\sqrt{5}\right)^n - \left(2-\sqrt{5}\right)^n}{2\sqrt{5}}$$

for $n = 1, 2, \ldots$. Then $\left(2+\sqrt{5}\right)^n = p_n + q_n\sqrt{5}, n = 0, 1, 2, \ldots$ and both obtained sequences satisfy the same recurrence

$$x_{n+1} = 4x_n + x_{n-1}, n \in \mathbb{N} \tag{1}$$

with initial conditions $p_0 = 1, p_1 = 2, q_0 = 0, q_1 = 1$. It is clear that $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are sequences of nonnegative integers and since

$$a_n + b_n\frac{1+\sqrt{5}}{2} = p_n + q_n\sqrt{5} \iff a_n + \frac{b_n}{2} + \frac{b_n}{2}\sqrt{5} = p_n + q_n\sqrt{5}$$

$$\iff \begin{cases} a_n + \dfrac{b_n}{2} = p_n \\ \dfrac{b_n}{2} = q_n \end{cases}$$

$$\iff \begin{cases} a_n = p_n - q_n \\ b_n = 2q_n \end{cases}, n \in \mathbb{N} \cup \{0\}$$

we have that $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are sequences of integers and can be defined independently by recurrence **(1)** with initial conditions $a_0 = 1, a_1 = 1, b_0 = 0, b_1 = 2$. In explicit form

$$b_n = \frac{\left(2+\sqrt{5}\right)^n - \left(2-\sqrt{5}\right)^n}{\sqrt{5}},\ a_n = \frac{\left(\sqrt{5}-1\right)\left(2+\sqrt{5}\right)^n + \left(\sqrt{5}+1\right)\left(2-\sqrt{5}\right)^n}{2\sqrt{5}}.$$

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Using Newton's binomial formula, exchanging $\sqrt{5}$ by $-\sqrt{5}$ in the first term results in exchanging $\sqrt{5}$ by $-\sqrt{5}$ in the second term, i.e., $(2+\sqrt{5})^n + (2-\sqrt{5})^n = 2a_n + b_n$ and $(2+\sqrt{5})^n - (2-\sqrt{5})^n = \sqrt{5}b_n$, yielding

$$b_n = \frac{(2+\sqrt{5})^n - (2-\sqrt{5})^n}{\sqrt{5}},\ a_n = \frac{(\sqrt{5}-1)(2+\sqrt{5})^n + (\sqrt{5}+1)(2-\sqrt{5})^n}{2\sqrt{5}}.$$