J165. Find all triples (x, y, z) of integers satisfying the system of equations

$$\begin{cases} \left(x^2+1\right)\left(y^2+1\right)+\frac{z^2}{10}=2010 \\ (x+y)(xy-1)+14z=1985. \end{cases}$$

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Note that $z = 10k$ for some integer k because $\dfrac{z^2}{10} = 2010 - \left(x^2+1\right)\left(y^2+1\right)$ is an integer. Let $p = x + y$ and $q = xy - 1$. Then

$$\left(x^2+1\right)\left(y^2+1\right) = x^2y^2 + x^2 + y^2 + 1 = (xy-1)^2 + (x+y)^2 = p^2 + q^2$$

and the system becomes

$$\begin{cases} p^2+q^2+10k^2=2010 \\ pq+140k=1985 \end{cases} \iff \begin{cases} p^2+q^2=2010-10k^2 \\ pq=1985-140k \end{cases} \qquad (1)$$

Since $(p-q)^2 = 2010 - 10k^2 - 2\,(1985 - 140k) = -10\,(k-14)^2$ then only $k = 14$ can provide solvability to (1). And for $k = 14$, (1) becomes $\begin{cases} p^2+q^2=50 \\ pq=25 \end{cases} \iff p = q = 5.$

Hence, $\begin{cases} x+y=5 \\ xy=4 \end{cases} \iff \begin{cases} x=4 \\ y=1 \end{cases}$ or $\begin{cases} x=1 \\ y=4 \end{cases}$ and triples $(5,1,140)\,,(1,5,140)$ are all integer solutions of the original system in integers.

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