J161. Let a, b, c be positive real numbers such that $a+b+c+2=abc$. Find the minimum of

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

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First solution by Arkady Alt, San Jose, California, USA

Note that $a+b+c+2=abc$ implies

$$3+2(a+b+c)+(ab+bc+ca)=abc+ab+bc+ca+a+b+c+1.$$

Then

$$(a+1)(b+1)(c+1)=\sum_{cyc}(a+1)(b+1)$$

and dividing by the left-hand side, we obtain

$$\sum_{cyc}\frac{1}{a+1}=1 \iff \sum_{cyc}\frac{1}{1+\dfrac{1}{a}}=2.$$

By the Cauchy-Schwarz inequality

$$\sum_{cyc}\left(1+\frac{1}{a}\right)\sum_{cyc}\frac{1}{1+\dfrac{1}{a}}\geq 9 \iff \sum_{cyc}\left(1+\frac{1}{a}\right)\geq\frac{9}{2} \iff \sum_{cyc}\frac{1}{a}\geq\frac{3}{2}$$

hence the minimum is $\frac{3}{2}$.

Second solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy

Let $x=\frac{1}{1+a}$, $y=\frac{1}{1+b}$, $z=\frac{1}{1+a}$ (see T.Andreescu, G.Dospinescu "Problems from the Book" XYZ Press, 2008). By trivial algebra we observe that $a+b+c+2=abc$ is equivalent to $x+y+z=1$ and then $a=\frac{1-x}{x}=\frac{y+z}{x}$, $b=\frac{1-y}{y}=\frac{x+z}{y}$, $c=\frac{1-z}{z}=\frac{y+x}{z}$. In terms of the variable x, y, z the inequality is

$$\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}, \qquad x+y+z=1$$

That the minimum of the above expression is $3/2$ is the content of Nesbitt's inequality which is well known. One of the many proof available is

$$\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y},\geq\frac{(x+y+z)^2}{2(xy+yz+zx)}=\frac{1}{2(xy+yz+zx)}\geq\frac{3}{2}$$

having employed Cauchy–Schwarz. Thus we have $xy+yz+zx\leq\frac{1}{3}$ which is obvious.

Also solved by Anthony Erb Lugo, San Juan, Puerto Rico; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Magkos Athanasios, Kozani, Greece; Jérôme Nicolas, Collège Versailles and Université Paul Cézanne (Faculté d'Économie Appliquée), France; Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Italy; Sayan Mukherjee, Kolkata, India.