J154. Let ABC be an acute triangle and let $MNPQ$ be a rectangle inscribed in the triangle such that $M, N \in BC, P \in AC, Q \in AB$. Prove that

$$\text{area} MNPQ \le \frac{1}{2} \text{area} ABC.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by Arkady Alt, San Jose, California, USA

Let $x = NP$ and $a = BC$. Since $PQ \parallel BC$ then $\triangle QAP \simeq \triangle BAC$ and, therefore,

$$\frac{PQ}{CB} = \frac{h_a - x}{h_a} \iff PQ = \frac{a\left(h_a - x\right)}{h_a}.$$

Hence,

$$\text{area} MNPQ = \frac{a}{h_a} \cdot x\left(h_a - x\right) \le \frac{a}{h_a} \cdot \left(\frac{x + \left(h_a - x\right)}{2}\right)^2 = \frac{a h_a}{4} = \frac{1}{2} \text{area} ABC.$$

Second solution by G.R.A.20 Problem Solving Group, Roma, Italy

Since ABC is an acute triangle then the rectangle $MNPQ$ is contained into the triangle ABC. Let AH be the height from A to BC, then

$$|QM| = |PN| = t|AH|, \ |BM| = t|BH|, \ |CN| = t|CH| \quad \text{for some } t \in [0, 1].$$

Therefore

$$\text{area}(MNPQ) = |QM|(|BC| - |BM| - |CN|) = t(1-t)|AH||BC| \le 2t(1-t)\text{area}(ABC).$$

The desired inequality follows by noting that

$$\min_{t \in [0,1]} 2t(1-t) = \frac{1}{2}.$$

Also solved by Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Raul A. Simon, Chile; Hoang Quoc Viet, Dang Huyen Trang, University of Auckland, New Zealand; Sayan Mukherjee, Kolkata, India.