J141. Let a, b, c be the side lengths of a triangle. Prove that

$$0 \le \frac{a-b}{b+c} + \frac{b-c}{c+a} + \frac{c-a}{a+b} < 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, "Babes-Bolyai" University, Romania

First solution by the authors. We can write

$$\sum_{cyc} \frac{a-b}{b+c} = \sum_{cyc} \frac{a+c}{b+c} - 3 = E - 3,$$

where

$$E = \frac{a+c}{b+c} + \frac{b+a}{b+c} + \frac{c+b}{b+c}.$$

For the right-hand side inequality observe that in any triangle we have $b+c > \frac{1}{2}(a+b+c)$, $c+a > \frac{1}{2}(a+b+c)$, and $a+b > \frac{1}{2}(a+b+c)$. It follows

$$E < \frac{2(a+c+b+a+c+b)}{a+b+c} = 4.$$

For the left-hand side inequality we use the Cauchy-Schwarz inequality and get

$$E = \sum_{cyc} \frac{a+c}{b+c} = \sum_{cyc} \frac{(a+c)^2}{(a+c)(b+c)} \ge \frac{(\sum_{cyc}(a+c))^2}{\sum_{cyc}(a+c)(b+c)} = \frac{4(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}.$$

The last fraction is greater than 3, since we have $a^2+b^2+c^2 \ge ab+bc+ca$. The equality holds if and only if the triangle is equilateral.

Second solution by Arkady Alt, San Jose, California, USA

Since

$$\begin{aligned}
\sum_{cyc} (a-b)(a+b)(c+a) &= \sum_{cyc} (a-b)\left(a^2+ab+bc+ca\right) \\
&= \sum_{cyc} (a-b)\,a^2 + (ab+bc+ca)\sum_{cyc}(a-b) \\
&= \sum_{cyc} (a-b)\,a^2 = a^3+b^3+c^3-a^2b-b^2c-c^2a \\
&= \frac{1}{3}\sum\left(2a^3+b^3-3a^2b\right) = \frac{1}{3}\sum (a-b)^2(2a+b)
\end{aligned}$$

then

$$\sum_{cyc} \frac{a-b}{b+c} = \frac{a^3+b^3+c^3-a^2b-b^2c-c^2a}{(a+b)(b+c)(c+a)} \geq 0.$$

It remains to prove that

$$\sum_{cyc} \frac{a-b}{b+c} < 1 \iff a^3+b^3+c^3-a^2b-b^2c-c^2a < (a+b)(b+c)(c+a).$$

We have

$$\begin{aligned}&(a+b)(b+c)(c+a)-a^3-b^3-c^3+a^2b+b^2c+c^2a\\&=2abc+a^2b+b^2c+c^2a+\sum_{cyc} a^2(b+c-a)) > 0\end{aligned}$$

because a, b, c satisfy the inequalities $b+c-a>0, c+a-b>0, a+b-c>0$.

Third solution by Michel Bataille, France

The central expression rewrites as $\frac{N}{D}$ with

$$N=(a-b)a^2+(b-c)b^2+(c-a)c^2 \quad \text{and} \quad D=(a+b)(b+c)(c+a).$$

Assume without loss of generality that $a=\max\{a,b,c\}$. Then,

$$N=(a-b)a^2+(b-c)b^2+(c-b)c^2+(b-a)c^2=(a-b)(a-c)(a+c)+(b-c)^2(b+c) \geq 0$$

and since $D>0$, we obtain $\frac{N}{D} \geq 0$.
It is easily checked that

$$D-N=a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)+a^2b+b^2c+c^2a+2abc,$$

hence $D-N>0$ (since a, b, c are the side lengths of a triangle, we have $a < b+c, b < c+a$ and $c < a+b$). Thus, $\frac{N}{D} < 1$ and we conclude that $0 \leq \frac{N}{D} < 1$, as required.

Also solved by G. C. Greubel, Newport News, USA; Ercole Suppa, Teramo, Italy; Daniel Lasaosa, Universidad Pública de Navarra, Spain.