J136. Let a, b, c be the sides, m_a, m_b, m_c the medians, h_a, h_b, h_c the altitudes, and l_a, l_b, l_c the angle bisectors of a triangle ABC. Prove that the diameter of the circumcircle of triangle ABC is equal to

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}}.$$

Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Bari, Italy

First solution by Arkady Alt, San Jose, California, USA

We should assume that $AB \neq AC$ because otherwise $l_a = h_a$ and in this case expression is undefined. Let AA_1 be angle bisector, then

$$\angle A_1AC = 90^\circ - \angle AA_1C = 90^\circ - B - \frac{A}{2} = \frac{C-B}{2}$$

and $h_a = l_a \cos\dfrac{C-B}{2}$. Hence, $\dfrac{l_a^2}{h_a^2} - 1 = \dfrac{1}{\cos^2\dfrac{C-B}{2}} - 1 = \tan^2\dfrac{C-B}{2}$. Since,

$$m_a^2 - h_a^2 = \frac{2\left(b^2+c^2\right)-a^2}{4} - \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{4a^2} = \frac{\left(b^2-c^2\right)^2}{4a^2}$$

then

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}} = \frac{l_a^2}{h_a^1}\sqrt{\frac{m_a^2 - h_a^2}{\dfrac{l_a^2}{h_a^2} - 1}} = \frac{1}{\cos^2\dfrac{C-B}{2}} \cdot \frac{\left|b^2-c^2\right|}{2a} \cdot \left|\cot\frac{C-B}{2}\right| = \frac{\left|b^2-c^2\right|}{a\left|\sin(B-C)\right|}$$

$$= \frac{4R^2\left|\sin^2 B - \sin^2 C\right|}{2R\sin A\left|\sin(B-C)\right|} = \frac{R\left|2\sin^2 B - 2\sin^2 C\right|}{\sin A\left|\sin(B-C)\right|} = \frac{R\left|\cos 2C - \cos 2B\right|}{\sin A\left|\sin(B-C)\right|}$$

$$= \frac{2R\left|\sin(B+C)\sin(B-C)\right|}{\sin A\left|\sin(B-C)\right|} = 2R.$$

Second solution by Aravind Srinivas L, Chennai, India

We know use the following relations: $l_a^2 = bc[1 - (\frac{a}{b+c})^2]$ $h_a = \frac{bc}{2R}$ $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$. We have to prove that

$$\frac{l_a^2}{h_a}\sqrt{\frac{m_a^2 - h_a^2}{l_a^2 - h_a^2}} = 2R$$

(R-circumradius of triangle ABC) Plugging the values of l_a^2, h_a, m_a^2 in the given to be proved equality, we have to prove after squaring both sides of the equality that

$$\frac{(b+c+a)^2(b+c-a)^2}{(b+c)^2} \cdot \frac{R^2(2b^2+2c^2-a^2) - b^2c^2((b^2+c^2)^2)}{4R^2a(b+c+a)(b+c-a) - b^2c^2((b+c)^2)} = 1$$