Junior problems

J133. A sequence $(a_n)_{n\geq 2}$ of real numbers greater than 1 satisfies the relation

$$a_n = \sqrt{1 + \frac{(n+1)!}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)}}$$

for all $n > 2$. Prove that if $a_k = k$ for some $k \geq 2$, then $a_n = n$ for all $n \geq 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Aravind Srinivas L, Chennai, India

Consider for some $n \in \mathbb{N}$ such that $n > 2$, we have by the definition of the sequence that : $a_n = \sqrt{1 + \frac{(n+1)!}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)}}$. Squaring both sides and subtracting by 1 on both sides, we get:

$$a_n^2 - 1 = \frac{(n+1)!}{2(a_2 - \frac{1}{a_2})\cdots(a_{n-1} - \frac{1}{a_{n-1}})} \qquad (1)$$

Now, consider a_{n+1}. Writing it according to the definition of the sequence, we have :

$$a_{n+1} = \sqrt{1 + \frac{(n+1)!(n+2)}{2\left(a_2 - \frac{1}{a_2}\right)\cdots\left(a_{n-1} - \frac{1}{a_{n-1}}\right)\left(a_n - \frac{1}{a_n}\right)}}.$$

Using (1) here, we get

$$a_{n+1}^2 = 1 + a_n(n+2) \qquad (2).$$

Now if $a_n = n$ for some $n \geq 2$, we can say by (2) that $a_{n+t} = n + t$ for $t \in \mathbb{N}$ Here, we take the sign of the sequence term (square root) depending upon the sign of the square root of the previous terms of the sequence as from (1) we know that $a_{n+t} > a_n$. This is because it is given that each of $a_n > 1$ in the first line of the problem statement and we use it here in this part, to say that sice it is given so, we can take only the positive sign of the value of the sequence for each term. In (2), if we replace n by $n-1$, we get $a_n = 1 + a_{n-1}(a_n + 1)$ which gives us easily that $a_{n-1} = \frac{n^2-1}{n+1} = n - 1$. And in this way, we can argue for all $n \geq 3$ and also thus get from $a_3 = 3$ that $a_2 = 2$. Thus, we have that $a_n = n$ for all $n \geq 2$, $n \in \mathbb{N}$. We have thus exhausted all the given conditions in the problem statement.

Second solution by Arkady Alt , San Jose ,California, USA

Let $b_n := \left(a_2 - \dfrac{1}{a_2}\right) \cdots \left(a_n - \dfrac{1}{a_n}\right), n \geq 2$, then for $n > 2$

$$a_n = \sqrt{1 + \frac{(n+1)!}{2b_{n-1}}} \iff a_n^2 - 1 = \frac{(n+1)!}{2b_{n-1}}$$

and, since

$$\frac{b_n}{b_{n-1}} = a_n - \frac{1}{a_n} = \frac{a_n^2 - 1}{a_n},$$

we have $\dfrac{b_n}{b_{n-1}} = \dfrac{(n+1)!}{2b_{n-1}a_n} \iff b_n = \dfrac{(n+1)!}{2a_n}$. Letting $b_{n-1} = \dfrac{n!}{2a_{n-1}}$ in $a_n^2 - 1 = \dfrac{(n+1)!}{2b_{n-1}}$ for $n > 2$ give us $a_n^2 - 1 = (n+1)\, a_{n-1}$. Thus, $a_{n+1}^2 = 1 + (n+2)\, a_n, n \geq 2$. Let $a_k = k$ for some $k \geq 2$. Then $a_n = n$ for any $n \geq k$. Indeed, since $a_k = k$ and in supposition $a_n = n, n \geq k$ we obtain $a_{n+1}^2 = 1 + (n+2)\, a_k = 1 + (n+2)\, n = (n+1)^2$ then by induction $a_n = n$ for any $n \geq k$ If $k > 2$ then for any $2 < n \leq k$ from supposition $a_n = n$ follows $a_{n-1} = \dfrac{a_n^2 - 1}{n+1} = \dfrac{n^2 - 1}{n+1} = n - 1$. Thus, by induction $a_n = n$ for any $2 \leq n \leq k$.

Also solved by G. C. Greubel, Newport News, USA; Daniel Lasaosa, Universidad Publica de Navarra, Spain