J126. Let a, b, c be positive real numbers. Prove that

$$3(a^2b^2 + b^2c^2 + c^2a^2)(a^2 + b^2 + c^2) \geq (a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2).$$

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First solution by Arkady Alt, San Jose, California, USA

Dividing original inequality by $a^2b^2c^2$ we obtain

$$3\left(a^2 + b^2 + c^2\right)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq \left(\frac{a}{b} + \frac{b}{a} + 1\right)\left(\frac{b}{c} + \frac{c}{b} + 1\right)\left(\frac{c}{a} + \frac{a}{c} + 1\right) \iff$$

(1) $9 + 3\sum_{cyc}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) \geq \prod_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) + \sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right) + \sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) + 1.$

Since

$$\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right) = \sum_{cyc}\left(\frac{a}{c} + \frac{c}{a} + \frac{b^2}{ca} + \frac{ca}{b^2}\right) = \sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) + \sum_{cyc}\left(\frac{a^2}{bc} + \frac{bc}{a^2}\right),$$

$$\prod_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) = 2 + \sum_{cyc}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)$$

then (1) $\iff$ $6 + 2\sum_{cyc}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) \geq 2\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) + \sum_{cyc}\left(\frac{a^2}{bc} + \frac{bc}{a^2}\right)$,

where latter inequality holds, because

$$6 + 2\sum_{cyc}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) - 2\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) - \sum_{cyc}\left(\frac{a^2}{bc} + \frac{bc}{a^2}\right) = \sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right)^2 +$$

$$\frac{1}{2}\sum_{cyc}\left(\frac{a^2}{b^2} + \frac{a^2}{c^2}\right) + \frac{1}{2}\sum_{cyc}\left(\frac{b^2}{a^2} + \frac{b^2}{c^2}\right) - 2\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) - \sum_{cyc}\left(\frac{a^2}{bc} + \frac{bc}{a^2}\right) =$$

$$\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{a}{b} + \frac{b}{a} - 2\right) + \frac{1}{2}\sum_{cyc}\left(\frac{a^2}{b^2} + \frac{a^2}{c^2} - \frac{2a^2}{bc}\right) + \frac{1}{2}\sum_{cyc}\left(\frac{b^2}{a^2} + \frac{b^2}{c^2} - \frac{2bc}{a^2}\right) =$$

$$\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right)\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \frac{1}{2}\sum_{cyc}\left(\frac{a}{b} - \frac{a}{c}\right)^2 + \frac{1}{2}\sum_{cyc}\left(\frac{b}{a} - \frac{b}{c}\right)^2 \geq 0.$$

Second solution by Gheorghe Pupazan, Chisinau, Republic of Moldova

After opening the brackets, the inequality becomes equivalent to:

$$2\sum a^2b^2(a^2 + b^2) + 6a^2b^2c^2 \geq abc(a^3 + b^3 + c^3) + \sum a^3b^3 + 2abc \cdot \sum ab(a + b).$$