J114. Let p be a prime. Find all solutions to the equation $a+b-c-d=p$, where a,b,c,d are positive integers such that $ab=cd$.

Proposed by Iurie Boreico, Harvard University, USA

First solution by John T. Robinson, Yorktown Heights, NY, USA

Let $x=ab=cd$, and let $y=\gcd(a,c)$, so that $a=ya'$ and $c=yc'$ (where a' and c' are relatively prime). Since $a|x$ and $c|x$ we can write

$$x=za'c'y\,.$$

Next, $b=x/a=zc'$ and $d=x/c=za'$. Substituting into $a+b-c-d=p$ we have:

$$ya'+zc'-yc'-za'=p\,;$$
$$(y-z)(a'-c')=p$$

or equivalently

$$(z-y)(c'-a')=p\,.$$

Since p is prime, an exhaustive list of possible solutions in positve integers y,z,a',c' is as follows:

$$y=z+1 \text{ and } a'=c'+p\,;$$
$$y=z+p \text{ and } a'=c'+1\,;$$
$$z=y+1 \text{ and } c'=a'+p\,;$$
$$z=y+p \text{ and } c'=a'+1\,.$$

These solutions all work out to be equivalent if, considering z (or y) and c' (or a') to be free parameters, we allow the resulting expressions for a and b, and c and d, to be switched. For example, considering z and c' to be parameters in the first solution above, we have $a=(z+1)(c'+p)$, $b=zc'$, $c=(z+1)c'$, $d=z(c'+p)$, and similarly for the other cases. All of the above solutions can be expressed in a more concise parametric form as follows, where u and v are any two positive integers:

$$\{a,\, b\}=\{uv,\ (u+1)(v+p)\}\,;$$
$$\{c,\, d\}=\{(u+1)v,\ u(v+p)\}\,.$$

Second solution by Arkady Alt, San Jose, California, USA

We will use the notation $x\perp y$ when integers x and y are relatively prime, i.e. $\gcd(x,y)=1$ and $x\mid y$ if x divides y. Let $s=\gcd(a,c)$. Since $a=ms$,

$c = sc_1$ then $\gcd(sm, sc_1) = s \iff \gcd(m, c_1) = 1$. Let $t = \gcd(b, c_1)$. Since $b = nt,\ \ c_1 = tc_2$ then $\gcd(tn, tc_2) = t \iff \gcd(n, c_2) = 1$. Thus we obtain $a = ms, b = tn, c = stc_2$, where $c_2 \perp n$ and $c_2 \perp m$ (because $c_2 \mid c_1$ and $c_1 \perp m$. Hence, $c_2 \perp mn$ and

$$\gcd(ab, c) = \gcd(smtn, stxc_2) = st\gcd(mn, c_2) = st$$

and, since $c \mid ab$ yields $\gcd(ab, c) = c$,then $st = c$. Let $a = ms, b = tn, c = st$. Then $ab = cd$ yields $d = mn$ and, therefore, $a + b - c - d = p \iff ms + nt - st - mn = p \iff (m - t)(s - n) = p$. Thus we have four types of solutions:

1. $\begin{cases} m - t = -1 \\ s - n = -p \end{cases}$ then $\begin{cases} m = t - 1 \\ n = s + p \end{cases}$ and

   $a = s(t - 1), b = t(s + p), c = st, d = (t - 1)(s + p);$

2. $\begin{cases} m - t = -p \\ s - n = -1 \end{cases}$ then $\begin{cases} m = t - p \\ n = s + 1 \end{cases}$ and

   $a = s(t - p), b = t(s + 1), c = st, d = (t - p)(s + 1);$

3. $\begin{cases} m - t = 1 \\ s - n = p \end{cases}$ then $\begin{cases} m = t + 1 \\ n = s - p \end{cases}$ and

   $a = s(t + 1), b = t(s - p), c = st, d = (s - p)(t + 1);$

4. $\begin{cases} m - t = p \\ s - n = 1 \end{cases}$ then $\begin{cases} m = t + p \\ n = s - 1 \end{cases}$ and

   $a = s(t + p), b = t(s - 1), c = st, d = (s - 1)(t + p),$

where s and t are any non-zero integers.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Assume without loss of generality that $a \ge b$ and $c \ge d$, since a, b are interchangeable, and so are c, d. Clearly $a > c \ge d > b$, since if $a = c$, then $b = d$ and $p = 0$ absurd, and similarly if $d = b$. Moreover, since $ab = cd$ but $a + b > c + d$, then $a - b > c - d$, and if $a \le c$, then $b < d$, yielding $ab < cd$ absurd. Clearly, $p(a + b + c + d) = (a + b)^2 - (b + d)^2 = (a + c)(a - c) + (b + d)(b - d)$, or

$$a + c = \frac{(b + d)(p + d - b)}{a - c - p} = b + d + \frac{p(b + d)}{d - b},$$

yielding $a - d = \frac{pd}{d-b}$ and $c - b = \frac{pb}{d-b}$.

Assume first that $d - b$ divides d and b. Hence, we may write $d - b = k$, and $d = (u + 1)k$, $b = uk$ for some positive integer u. Then, $a = (u + 1)(k + p)$ and $c = u(k + p)$. Note that these necessary forms for a, b, c, d satisfy both conditions, since $ab = cd = u(u + 1)k(k + p)$. All possible solutions in this case