OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(5), p. 194–195.

OC181. All the prime numbers are written in order $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on. Find all pairs of positive integers a and b with $a - b \geq 2$ such that $p_a - p_b$ divides $2(a - b)$.

Originally problem 1 from the 2013 Mexico National Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

The pair

$$a = 4, \quad b = 2$$

is a solution, since $p_4 = 7$ and $p_2 = 3$ and $p_4 - p_2 = 4$ is a divisor of $2(4 - 2) = 4$. We show that there are no other solutions.

Suppose (a, b) is any solution. If $b = 1$ then $p_b = 2$, and $p_a - p_b$ is an odd number that divides $a - b$. Moreover, $p_a \geq 2a - 1$ and $a > 2$. Hence

$$a - b \geq p_a - p_b \geq 2a - 3 > a - 1 = a - b,$$

a contradiction. Thus the numbers p_a and p_b are odd primes, which implies that $p_a - p_b \geq 2(a - b)$. We obtain $p_a - p_b = 2(a - b)$ so that all odd numbers between p_b and p_a are primes. Then, the numbers

$$p_b, \quad p_b + 2, \quad p_b + 4 \tag{3}$$

are primes. But one of the numbers (1) is divisible by 3. It follows $p_b = 3$. Since $a - b \geq 2$ and all odd numbers between 3 and p_a are primes, we have $p_a = 7$.

OC182. Let x and y be real numbers satisfying $x^2y^2 + 2yx^2 + 1 = 0$. If

$$S = \frac{2}{x^2} + 1 + \frac{1}{x} + y\left(y + 2 + \frac{1}{x}\right),$$

find the maximum and minimum values of S.

Originally problem 2 from the 2013 Uzbekistan National Olympiad.

We received five correct submissions and one incorrect submission. We present the solution by Arkady Alt.

Note that

$$\begin{aligned} x^2y^2 + 2yx^2 + 1 = 0 &\iff x^2y^2 + 2yx^2 + x^2 + 1 - x^2 = 0 \\ &\iff x^2(y+1)^2 + 1 - x^2 = 0 \\ &\iff (y+1)^2 + \frac{1}{x^2} = 1 \end{aligned}$$

This implies that there is a real number $t \neq \frac{(2n+1)\pi}{2}$ with $y+1 = \sin t$ and $\frac{1}{x} = \cos t$. Further,

$$\begin{aligned} S &= \frac{2}{x^2} + \left(2 + \frac{1}{x} + y\right) - (y+1) + y\left(y + 2 + \frac{1}{x}\right) \\ &= \frac{2}{x^2} + (y+1)\left(y + 1 + \frac{1}{x}\right) \\ &= \frac{2}{x^2} + (y+1)^2 + (y+1)\cdot\frac{1}{x} \\ &= 1 + \frac{1}{x^2} + \frac{1}{x}(y+1) \end{aligned}$$

Combining the above information yields

$$S = 1 + \cos^2 t + \sin t \cos t = \frac{3 + \cos 2t + \sin 2t}{2} = \frac{3 + \sqrt{2}\sin(2t + \pi/4)}{2}$$

and, therefore,

$$S_{\max} = \frac{3+\sqrt{2}}{2}, \qquad S_{\min} = \frac{3-\sqrt{2}}{2}.$$

Note that these values are actually obtained. For $t^* = \pi/8$ we have

$$(x^*, y^*) = \left(\frac{1}{\cos\pi/8}, \sin\pi/8 - 1\right),$$

and hence $S(x^*, y^*) = \dfrac{3+\sqrt{2}}{2}$. On the other hand, for $t_* = -3\pi/8$, we have

$$(x_*, y_*) = \left(\frac{1}{\cos(-3\pi/8)}, \sin(-3\pi/8) - 1\right),$$

and hence $S(x_*, y_*) = \dfrac{3-\sqrt{2}}{2}$, since

$$\cos\pi/8 = \frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \sin\pi/8 = \cos(-3\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2},$$
$$\sin(-3\pi/8) = -\cos\pi/8 = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

OC183. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(0) = 0$, $f(1) = 2013$ and

$$(x-y)(f(f(x)^2) - f(f(y)^2)) = (f(x) - f(y))(f(x)^2 - f(y)^2).$$

Originally problem 1 from day 2 of the 2013 Vietnam National Olympiad.

We received two correct submissions. We present the solution by Michel Bataille.