4857. *Proposed by Toyesh Prakash Sharma.*

Let a, b, c be positive real numbers such that $a+b+c=\dfrac{3}{2}$. Show that

$$a^a b^b + b^b c^c + c^c a^a \geq \frac{3}{2}.$$

We received 18 submissions, all correct and complete. We present two solutions, slightly altered by the editor.

Solution 1, by Arkady Alt.

By the AM-GM inequality we have

$$a^a b^b + b^b c^c + c^c a^a \geq 3\sqrt[3]{a^a b^b \cdot b^b c^c \cdot c^c a^a} = 3\left(a^a b^b c^c\right)^{2/3}.$$

Also, by the weighted AM-GM inequality

$$\frac{1}{a^a b^b c^c} = \left(\frac{1}{a}\right)^a \left(\frac{1}{b}\right)^b \left(\frac{1}{c}\right)^c \leq \left(\frac{a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c}}{a+b+c}\right)^{a+b+c} = \left(\frac{3}{a+b+c}\right)^{a+b+c},$$

which is equivalent to each of:

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c} = \left(\frac{1}{3} \cdot \frac{3}{2}\right)^{3/2} = \left(\frac{1}{2}\right)^{3/2} \quad \text{and} \quad \left(a^a b^b c^c\right)^{2/3} \geq \frac{1}{2}.$$

Hence,

$$a^a b^b + b^b c^c + c^c a^a \geq 3\left(a^a b^b c^c\right)^{2/3} \geq \frac{3}{2}.$$

Solution 2, by Michel Bataille.

The functions $f(x) = x \ln x$ and $g(x) = x^x$ are convex on $(0, \infty)$ (since $f''(x) = \frac{1}{x}$ and $g''(x) = \left(\frac{1}{x} + (1+\ln x)^2\right) x^x$ are positive on $(0, \infty)$). We deduce that

$$a^a b^b = e^{f(a)+f(b)} \geq e^{2f((a+b)/2)} = (g((a+b)/2))^2$$

and therefore

$$a^a b^b + b^b c^c + c^c a^a \geq (g((a+b)/2))^2 + (g((b+c)/2))^2 + (g((c+a)/2))^2.$$

Now, using the fact that the function $x \mapsto x^2$ is increasing and convex on $(0, \infty)$ and the convexity of g, we obtain

$$\begin{aligned} a^a b^b + b^b c^c + c^c a^a &\geq 3\left(\frac{g((a+b)/2) + g((b+c)/2) + g((c+a)/2)}{3}\right)^2 \\ &\geq \frac{1}{3}\left(3g\left(\frac{a+b+c}{3}\right)\right)^2 = 3\left(g(1/2)\right)^2 = \frac{3}{2}. \end{aligned}$$

Editor's Comment. Most solutions did not mention that equality occurs only when $a = b = c$.