4198. *Proposed by Leonard Giugiuc.*

In a triangle ABC, we have that $\sin A \sin B \sin C = \frac{2+\sqrt{3}}{8}$. Find the maximum possible value of $\cos A \cos B \cos C$.

We received three correct solutions and one incorrect solution. We present the solution by Arkady Alt.

Since

$$\sin^2 75^\circ = \frac{1-\cos 150^\circ}{2} = \frac{1+\cos 30^\circ}{2} = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4}$$

and

$$\cos^2 75^\circ = 1 - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4},$$

then for $A = 30^\circ$ and $B = C = 75^\circ$ we have

$$\sin A \sin B \sin C = \sin 30^\circ \sin^2 75^\circ = \frac{2+\sqrt{3}}{8}$$

and

$$\cos A \cos B \cos C = \cos 30^\circ \cos^2 75^\circ = \frac{2-\sqrt{3}}{4} \cdot \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}-3}{8}.$$

We will prove that for any $A, B, C > 0$ such that $\sin A \sin B \sin C = \frac{2+\sqrt{3}}{8}$ and $A + B + C = \pi$, we have

$$\cos A \cos B \cos C \leq \frac{2\sqrt{3}-3}{8}, \qquad (1)$$

with equality if $A = 30^\circ$ and $B = C = 75^\circ$ and permutations of this.

By the symmetry of inequality (1), we can assume that $C = \max\{A, B, C\}$, and hence that $C \geq \frac{\pi}{3}$.

We can also assume that $C < \frac{\pi}{2}$ since otherwise $\cos A \cos B \cos C \leq 0$. Since

$$\cos A \cos B - \sin A \sin B = \cos(A+B) = -\cos C$$

or, equivalently

$$\cos A \cos B = \sin A \sin B - \cos C$$

and

$$\sin A \sin B = \frac{2+\sqrt{3}}{8},$$

then

$$\cos A \cos B \cos C = \left(\frac{2+\sqrt{3}}{8 \sin C} - \cos C\right) \cos C = \frac{2+\sqrt{3}}{8t} - \frac{1}{1+t^2},$$

where $t = \tan C \geq \sqrt{3}$. Inequality (1) becomes

$$\frac{2+\sqrt{3}}{8t} - \frac{1}{1+t^2} \leq \frac{2\sqrt{3}-3}{8}. \qquad (2)$$

Let $k = 2+\sqrt{3}$ and $h(t) = \frac{k}{8t} - \frac{1}{1+t^2}$. Then $k^2 - 4k + 1 = 0$ and

$$h(k) = \frac{1}{8} - \frac{1}{1+k^2} = \frac{1}{8} - \frac{1}{4k} = \frac{k-2}{8k} = \frac{\sqrt{3}}{8(2+\sqrt{3})} = \frac{2\sqrt{3}-3}{8}.$$

Thus inequality (2) can be rewritten as $h(t) \leq h(k)$ for $t \geq \sqrt{3}$. We have

$$\begin{aligned} h(k) - h(t) &= \frac{1}{8} - \frac{1}{1+k^2} - \frac{k}{8t} + \frac{1}{1+t^2} = \frac{t-k}{8t} - \frac{t^2-k^2}{(k^2+1)(t^2+1)} \\ &= (t-k)\left(\frac{1}{8t} - \frac{t+k}{4k(t^2+1)}\right) = \frac{(t-k)[k(t-1)^2 - 2t^2]}{8kt(t^2+1)} \\ &= \frac{(t-k)^2[t(k-2)-1]}{8kt(t^2+1)} \geq 0 \end{aligned}$$

because $k - 2 = \sqrt{3}$ and $t \geq \sqrt{3}$. Hence,

$$\max \cos A \cos B \cos C = \max_{t>\sqrt{3}} h(t) = h(k) = \frac{2\sqrt{3}-3}{8}.$$

4199. *Proposed by Michel Bataille.*

Let two circles Γ_1, Γ_2, with respective centres O_1, O_2, intersect at A and B and let ℓ be the internal bissector of $\angle O_1AO_2$. Let $M_1, M_2 \neq A, B$ be points on Γ_2. For $k = 1, 2$, the line BM_k and the reflection of AM_k in ℓ intersect Γ_1 again at N_k and P_k , respectively. Prove that $N_1P_2 = N_2P_1$.

All three submissions were correct, but only Steven Chow, whose solution we feature, established the generalization.

The restriction on the line ℓ can be relaxed; specifically, the following argument shows that the result holds for an arbitrary line ℓ through A. We use directed angles (mod π).

Note that AM_1 and AP_1 are symmetric about ℓ, as are AM_2 and AP_2. Therefore, $\angle P_2AP_1 = \angle M_1AM_2$. Because $\angle M_1AM_2 = \angle M_1BM_2$, we therefore have

$$\angle P_2AP_1 = \angle M_1BM_2 = \angle N_1BN_2.$$

Consequently,

$$\angle N_2AP_1 = \angle N_2AP_2 + \angle P_2AP_1 = \angle N_2BP_2 + \angle N_1BN_2 = \angle N_1BP_2.$$

It follows that $\angle N_1BP_2$ and $\angle N_2AP_1$ are subtended by equal chords, namely $N_1P_2 = N_2P_1$.