4144. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=1$. Find the maximum value of the expression

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3.$$

We received 19 submissions, all of which are correct. We present two solutions with the first one being a composite of very similar solutions by several solvers.

Solution 1, by Arkady Alt, Michel Bataille, Steven Chow, and Daniel Dan (independently).

Let $q=ab+bc+ca$ and $r=abc$. Then

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=1-2q$$

and

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc=1-3q+3r.$$

Hence,

$$\begin{aligned}&\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3\\&=a^3+b^3+c^3-\frac{3}{2}(a^2+b^2+c^2)+\frac{3}{4}(a+b+c)-\frac{3}{8}\\&=1-3q+3r-\frac{3}{2}(1-2q)+\frac{3}{8}=3r-\frac{1}{8}=3abc-\frac{1}{8}\\&\le 3\left(\frac{a+b+c}{3}\right)^3-\frac{1}{8}=\frac{1}{9}-\frac{1}{8}=-\frac{1}{72}.\end{aligned}$$

Therefore, the required maximum $-\frac{1}{72}$ is attained exactly when $a=b=c=\frac{1}{3}$.

Solution 2, by Titu Zvonaru.

We prove that the searched maximum is $-\frac{1}{72}$.

For convenience, let $d=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$. Then

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3\le-\frac{1}{72}$$

is equivalent in succession to

$$\begin{aligned}&9((2a-1)^3+(2b-1)^3+(2c-1)^3)\le 1,\\&9((a-b-c)^3+(b-c-a)^3+(c-a-b)^3)\le-(a+b+c)^3,\\&-9(a^3+b^3+c^3)-27d+162abc\le-(a^3+b^3+c^3)-3d-6abc,\\&8(a^3+b^3+c^3)+24d\ge 168abc,\\&a^3+b^3+c^3+3d\ge 21abc,\end{aligned}$$

which is true since by the AM-GM inequality we have

$$a^3 + b^3 + c^3 + 3d \geq 3abc + 3(6abc) = 21abc.$$

4145. *Proposed by Leonard Giugiuc.*

Prove that the system

$$\begin{cases} A^3 + A^2B + AB^2 + ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \\ \\ B^3 + B^2A + BA^2 + BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{cases}$$

has no solutions in the set of 3×3 matrices over complex numbers.

We received 11 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Assume by contradiction that there exist matrices A and B that are solutions to this system.

$$\begin{cases} A^3 + A^2B + AB^2 + ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, & (1) \\ \\ B^3 + B^2A + BA^2 + BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. & (2) \end{cases}$$

Equation (1) gives

$$A(A+B)^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det(A) \cdot \det(A+B)^2 = 1$$

$$\Rightarrow \det(A+B) \neq 0. \qquad (3)$$

From equations (1) and (2), we get

$$(A+B)^3 = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \det(A+B)^3 = 0 \Rightarrow \det(A+B) = 0,$$

which contradicts (3).