and so

$$[A_0A_1\dots A_{n-1}]=\sum_{i=1}^{n}[A_{i-1}TA_i]\le\sum_{i=1}^{n}[A_{i-1}\Omega_iA_i]=\sum_{i=1}^{n}\frac{1}{n}[\Pi_i]$$

and the required inequality follows.

4142. *Proposed by Daniel Sitaru.*

Prove that if $a,b,c\in(0,\infty)$ then:

$$\left(1+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}}\le\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right).$$

We received 4 correct solutions. We present the solution by Arkady Alt.

Assuming, due to the homogeneity of the original inequality, that $a+b+c=1$ and denoting

$$p=ab+bc+ca,\ \ q=abc,$$

we obtain

$$a^2+b^2+c^2=1-2p,$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)=\frac{(a+b)\,(b+c)\,(c+a)}{abc}=\frac{p-q}{q},$$

and

$$1+\frac{a^2+b^2+c^2}{ab+bc+ca}=1+\frac{1-2p}{p}=\frac{1-p}{p}.$$

The original inequality thus becomes

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}}\le\frac{p}{q}-1.$$

Since $0<q\le\dfrac{p^2}{3}$, we have $\dfrac{p}{q}\ge\dfrac{3}{p}$, and it suffices to prove the inequality

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}}\le\frac{3}{p}-1.$$

For $0<p\le\frac{1}{3}$, this is successively equivalent to

$$\frac{1-p}{p}\le\left(\frac{3-p}{p}\right)^{1-2p},$$

$$\left(\frac{3-p}{p}\right)^{2p}\le\frac{3-p}{1-p},$$

$$\left(\frac{3}{p}-1\right)^2\le\left(\frac{\frac{3}{p}-1}{\frac{1}{p}-1}\right)^{\frac{1}{p}}.$$

Denoting $t = \frac{1}{p} \in [3, \infty)$, we obtain the following more convenient equivalent form of the latter inequality.

$$(3t-1)^2 \le \left(\frac{3t-1}{t-1}\right)^t \iff t \ln\left(\frac{3t-1}{t-1}\right) \ge 2\ln(3t-1).$$

Let

$$h(t) = t\left[\ln(3t-1) - \ln(t-1)\right] - 2\ln(3t-1).$$

Then

$$\begin{aligned} h'(t) &= \ln(3t-1) - \ln(t-1) + t\left(\frac{3}{3t-1} - \frac{1}{t-1}\right) - \frac{6}{3t-1} \\ &= \ln(3t-1) - \ln(t-1) - \frac{1}{t-1} - \frac{5}{3t-1} \end{aligned}$$

and

$$h''(t) = \frac{3}{3t-1} - \frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{15}{(3t-1)^2} = \frac{2\left(9t^2 - 14t + 7\right)}{(3t-1)^2 (t-1)^2}.$$

Since $h''(t) > 0$ for $t \ge 3$, $h'(t)$ increases on $[3, \infty)$ and, therefore,

$$h'(t) \ge h'(3) = \ln 8 - \ln 2 - \frac{1}{2} - \frac{5}{8} = 2\ln 2 - \frac{9}{8} > 0.$$

Hence, $h(t)$ increases on $[3, \infty)$ and, therefore,

$$h(t) \ge h(3) = 3(\ln 8 - \ln 2) - 2\ln 8 = 0.$$

Thus, $t \ln\left(\dfrac{3t-1}{t-1}\right) \ge 2\ln(3t-1)$, as desired.

4143. *Proposed by Roy Barbara.*

For any real number $x \ge 1$, let $y = x^{1/2} + x^{-1/2}$.

a) Express x in terms of y by a radical formula and check that no rational fraction $F(t)$ can exist such that $x = F(y)$. (A rational fraction is an expression of the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials with rational coefficients.)

b) Find a closed form formula $x = F(y)$ containing no radicals.

c) $\star$ Is there a *complex* fraction such that $x = F(y)$? (A complex fraction is a function of the form $f(z)/g(z)$, where $f(t)$ and $g(t)$ are polynomials with complex coefficients.)

We received four solutions, all correct, and feature that of Joseph DiMuro.

a) We have $y = \frac{x+1}{\sqrt{x}}$, which can be rewritten as $x - y\sqrt{x} + 1 = 0$. The quadratic formula then gives us $\sqrt{x} = \frac{y \pm \sqrt{y^2-4}}{2}$. Both of these possible expressions for $\sqrt{x}$