Solution 2, by Steven Chow.

Modulo a^2+b^2, we have that $b^2 \equiv -a^2$, $2a^3-a^2\equiv 0$, and

$$2a^3b^2+ab^2+3b^4 \equiv -2a^5-a^3+3a^4 = (2a^3-a^2)(a-a^2)\equiv 0.$$

Editor's Comments. The condition is, in fact, satisfiable. For example, we can take

$$(a,b)=(2c^2+1, 2c(2c^2+1))$$

for any nonzero integer c and find that

$$a^2+b^2=(2c^2+1)^2(4c^2+1)$$

and

$$2a^3+b^2=2(a^2+b^2).$$

Are there other integer pairs that will work?

4133. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Consider the sequence (a_n) defined recursively by $a_1=1$ and $a_{n+1}=(2n+1)!!a_n$ for all positive integers n. Compute

$$\lim_{n\to\infty}\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}.$$

We received five solutions, all correct and complete. We present two solutions.

Solution 1, by Arkady Alt.

We will use convenient asymptotic notation, namely in the case $\lim\limits_{n\to\infty}\dfrac{b_n}{c_n}=1$ we will write $b_n\sim c_n$ and then $\lim\limits_{n\to\infty}x_nb_n=\lim\limits_{n\to\infty}x_nc_n$ for any convergent sequence x_n.

Since $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}=\dfrac{1}{e}$ then $\sqrt[n]{n!}\sim\dfrac{n}{e}$ and therefore

$$\sqrt[2n]{(2n-1)!!}=\sqrt[2n]{\frac{(2n)!}{2^nn!}}=\frac{1}{\sqrt{2}}\sqrt[2n]{\frac{(2n)!}{n!}}\sim\frac{1}{\sqrt{2}}\cdot\frac{\frac{2n}{e}}{\sqrt{\frac{n}{e}}}=\sqrt{\frac{2n}{e}}$$

$$\iff \sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}.$$

Hence, $\lim\limits_{n\to\infty}\dfrac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}=\sqrt{\dfrac{2}{e}}\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{\sqrt[n^2]{a_n}}.$

Let $c_n:=\ln\dfrac{\sqrt{n}}{\sqrt[n^2]{a_n}}=\dfrac{\ln n}{2}-\dfrac{\ln a_n}{n^2}=\dfrac{d_n}{2n^2}$, where $d_n:=n^2\ln n-2\ln a_n$

Note that

$$\begin{aligned}
&d_{n+1}-d_n\\
&=(n+1)^2\ln(n+1)-n^2\ln n-2\left(\ln a_{n+1}-\ln a_n\right)\\
&=(n+1)^2\ln(n+1)-n^2\ln n-2\ln(2n+1)!!\\
&=\Big((n+1)^2\ln(n+1)-n^2\ln(n+1)\Big)+\Big(n^2\ln(n+1)-n^2\ln n\Big)-2\ln(2n+1)!!\\
&=(2n+1)\ln(n+1)+n^2\ln\left(1+\frac{1}{n}\right)-2\ln(2n+1)!!\\
&=\ln(n+1)+n^2\ln\left(1+\frac{1}{n}\right)+2n\ln(n+1)-2\ln(2n+1)!!\\
&=\ln(n+1)+n\ln\left(1+\frac{1}{n}\right)^n+2n\cdot\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{d_{n+1}-d_n}{(n+1)^2-n^2}\\
&\quad=\lim_{n\to\infty}\frac{\ln(n+1)+n\ln\left(1+\frac{1}{n}\right)^n+2n\cdot\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}}{2n+1}\\
&\quad=\lim_{n\to\infty}\frac{\ln(n+1)}{2n+1}+\lim_{n\to\infty}\frac{n\ln\left(1+\frac{1}{n}\right)^n}{2n+1}+\lim_{n\to\infty}\frac{2n\cdot\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}}{2n+1}\\
&\quad=\frac{1}{2}+\lim_{n\to\infty}\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}\\
&\quad=\frac{1}{2}+\ln\lim_{n\to\infty}\frac{n+1}{\sqrt[n]{(2n+1)!!}}\\
&\quad=\frac{1}{2}+\ln\frac{e}{2}\\
&\quad=\frac{3}{2}-\ln 2.
\end{aligned}$$

(Observe that

$$\lim_{n\to\infty}\frac{n+1}{\sqrt[n]{(2n+1)!!}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{n+1}{n}\cdot\frac{1}{\sqrt[n]{(2n+1)}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\frac{e}{2}.)$$

Then by Stolz-Cesaro Theorem $\lim_{n\to\infty}\frac{d_n}{n^2}=\frac{3}{2}-\ln 2$ and therefore

$$\lim_{n\to\infty}c_n=\frac{1}{2}\left(\frac{3}{2}-\ln 2\right).$$

Hence, $\lim_{n\to\infty}\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}=\sqrt{\frac{2}{e}}\cdot e^{\lim_{n\to\infty}c_n}=\sqrt{\frac{2}{e}}\cdot e^{\frac{3}{4}-\frac{1}{2}\ln 2}=e^{\frac{1}{4}}$.