4092. *Proposed by Mihaela Berindeanu.*

Show that

$$\left[\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right]\left[\frac{b^2+16b+80}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right]\geq\frac{9}{4}$$

for all $a, b > 0$. When does equality hold?

We received ten correct submissions. We present two solutions.

Solution 1, by Arkady Alt.

Since

$$\frac{a^2+16a+80}{16\,(a+4)}=1+\frac{a^2+4^2}{16\,(a+4)},$$

we have

$$\begin{aligned}&\left(\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right)\left(\frac{b^2+16b+80}{16(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right)\\&=\left(1+\frac{a^2+4^2}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+4^2)}}\right)\left(1+\frac{b^2+4^2}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+4^2)}}\right)\end{aligned}$$

and, combining Cauchy-Schwarz Inequality and inequality $\sqrt{2\,(u^2+v^2)}\geq u+v$, we obtain

$$\begin{aligned}&\left(1+\frac{a^2+4^2}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+4^2)}}\right)\left(1+\frac{2}{\sqrt{2\,(a^2+4^2)}}+\frac{b^2+4^2}{16\,(b+4)}\right)\\&\geq\left(1\cdot 1+\sqrt{\frac{a^2+4^2}{16\,(a+4)}}\cdot\sqrt{\frac{2}{\sqrt{2\,(a^2+4^2)}}}+\sqrt{\frac{2}{\sqrt{2\,(b^2+4^2)}}}\cdot\sqrt{\frac{b^2+4^2}{16\,(b+4)}}\right)^2\\&=\left(1+\frac{1}{4}\sqrt{\frac{\sqrt{2\,(a^2+4^2)}}{a+4}}+\frac{1}{4}\sqrt{\frac{\sqrt{2\,(b^2+4^2)}}{b+4}}\right)^2\\&\geq\left(1+\frac{1}{4}+\frac{1}{4}\right)^2=\frac{9}{4}.\end{aligned}$$

Since in inequality $\sqrt{2\,(u^2+v^2)}\geq u+v$ equality occurs if and only if $u=v$, it is easy to see that the equality holds if and only if $a=b=4$.

Solution 2, by AN-anduud Problem Solving Group.

Using AM-GM inequality, we get

$$\sqrt{2(b^2+16)}=\sqrt{8\cdot\frac{b^2+16}{4}}\leq\frac{1}{2}\left(8+\frac{b^2+16}{4}\right)=\frac{b^2+48}{8}. \tag{1}$$

Applying AM-GM inequality and using (1), we have