4085. Proposed by José Luis Díaz-Barrero. Correction.

Let ABC be an acute triangle. Prove that

$$\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}<\frac{3\sqrt{2}}{2}.$$

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We will prove that if ABC is an acute triangle then maximal value of $\sum_{cyc}\sqrt[4]{\sin(\cos A)\cdot\cos B}$ *is* $3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$, that is

(**1**) $\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}\leq 3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$

with equality iff $A=B=C=\frac{\pi}{3}$.

Proof.

By Cauchy Inequality $\sum_{cyc}\sqrt[4]{\sin(\cos A)\cdot\cos B}\leq\sqrt{\sum_{cyc}\sqrt{\sin(\cos A)}}\cdot\sqrt{\sum_{cyc}\sqrt{\cos A}}$.

Since $\sqrt{\sin x}$ and $\sqrt{\cos x}$ both are concave down on $(0,\pi/2)$

$\left(\left(\sqrt{\sin x}\right)''=-\frac{1+\sin^2 x}{4\sin x\cdot\sqrt{\sin x}}<0,\ \left(\sqrt{\cos x}\right)''=-\frac{1+\cos^2 x}{4\cos x\cdot\sqrt{\cos x}}<0, x\in(0,\pi/2)\right)$

then by Jensen's Inequality we have

$$\sum_{cyc}\sqrt{\sin(\cos A)}\leq 3\sqrt{\sin\left(\frac{1}{3}\sum_{cyc}\cos A\right)}\leq 3\sqrt{\sin\frac{1}{2}}$$

($\sum_{cyc}\cos A\leq\frac{3}{2}$ because $\sum_{cyc}\cos A=1+\frac{r}{R}$, $2r\leq R$ (Euler Inequality))

and

$$\sum_{cyc}\sqrt{\cos A}\leq 3\sqrt{\cos\frac{A+B+C}{3}}=3\sqrt{\cos\frac{\pi}{3}}=3\sqrt{\frac{1}{2}}.$$

Hence, $\sqrt{\sum_{cyc}\sqrt{\sin(\cos A)}}\cdot\sqrt{\sum_{cyc}\sqrt{\cos A}}\leq\sqrt{3\sqrt{\sin\frac{1}{2}}}\cdot\sqrt{3\sqrt{\frac{1}{2}}}\leq 3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$ and, therefore,

$$\sum_{cyc}\sqrt[4]{\sin(\cos A)\cdot\cos B}\leq 3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}.$$

Thus, maximal value of $\sum_{cyc}\sqrt[4]{\sin(\cos A)\cdot\cos B}$ when ABC is an acute triangle is $3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$

Remark.

Since $\sin\frac{1}{2}<\frac{1}{2}$ then $\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}<\sqrt{\frac{1}{2}}$ and, therefore,

$$\sum_{cyc}\sqrt[4]{\sin(\cos A)\cdot\cos B}\leq 3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}<\frac{3\sqrt{2}}{2}.$$