that joins those end points. Consequently, we have

$$\begin{aligned}\frac{1}{4}\left((f(2A)+f(2B)+f(2C)\right) &\geq \frac{1}{4}\left((g(2A)+g(2B)+g(2C)\right)\\ &= \frac{1}{4}\left(-\frac{2(2A+2B+2C)}{\pi}+6\right)=\frac{1}{2}.\end{aligned}$$

We conclude that the product $\sin A \sin B \sin C$ cannot be less than $\frac{1}{2}$ when all three angles are restricted to the domain $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. The minimum is achieved if and only if $f(x)=g(x)$ for x equal to $2A, 2B$, and $2C$; because $A+B+C=\pi$, this is possible only if one of the angles is $\frac{\pi}{2}$ while the other two are $\frac{\pi}{4}$.

4062. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let L_n denote the nth Lucas number defined by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Prove that

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\geq \frac{2}{3}L_{n+4}^2.$$

We received ten correct and complete solutions. We present the solutions of Arkady Alt, who like most submitters used standard inequalities for a simple proof, and a slightly modified version of the solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, who made heavier use of the given recursion to find a stronger bound.

Solution 1, by Arkady Alt.

Since $a^4+b^4 \geq ab(a^2+b^2)$ (as this can be rewritten as $(a^2+ab+b^2)(a-b)^2 \geq 0$) and $a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3}$ for all $a, b, c \in \mathbb{R}$, we have

$$\begin{aligned}&\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}\\ &\geq \frac{L_nL_{n+1}(L_n^2+L_{n+1}^2)}{L_nL_{n+1}}+\frac{L_{n+1}L_{n+3}(L_{n+1}^2+L_{n+3}^2)}{L_{n+1}L_{n+3}}+\frac{L_{n+3}L_n(L_{n+3}^2+L_n^2)}{L_{n+3}L_n}\\ &= 2(L_n^2+L_{n+1}^2+L_{n+3}^2)\\ &\geq 2\frac{(L_n+L_{n+1}+L_{n+3})^2}{3}=\frac{2(L_{n+2}+L_{n+3})^2}{3}\\ &= \frac{2L_{n+4}^2}{3}\end{aligned}$$

Solution 2, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

More generally, we will show that for all $n \geq 0$,

$$\frac{L_n^4+L_{n+1}^4}{L_nL_{n+1}}+\frac{L_{n+1}^4+L_{n+3}^4}{L_{n+1}L_{n+3}}+\frac{L_{n+3}^4+L_n^4}{L_{n+3}L_n}>2L_{n+4}^2.$$

We can check by hand that this holds for $n \leq 2$.

For $n \geq 3$ we first use the Arithmetic Mean - Geometric Mean inequality to obtain

$$\begin{aligned}x^4 + y^4 &= 2x^2y^2 + (x^2 - y^2)^2\\ &= 2x^2y^2 + (x+y)^2(x-y)^2\\ &\geq 2x^2y^2 + 4xy(x-y)^2\\ &= xy(2xy + 4(x-y)^2)\end{aligned}$$

and hence

$$\frac{x^4 + y^4}{xy} \geq 2xy + 4(x-y)^2.$$

Using this property and the recursion for the Lucas numbers (multiple times, when necessary), we get

$$\frac{L_n^4 + L_{n+1}^4}{L_nL_{n+1}} \geq 2L_nL_{n+1} + 4(L_{n+1} - L_n)^2 = 4L_{n+1}^2 - 6L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1}L_{n+3}} \geq 2L_{n+1}L_{n+3} + 4(L_{n+3} - L_{n+1})^2 = 8L_{n+1}^2 + 10L_{n+1}L_n + 4L_n^2,$$

$$\frac{L_{n+3}^4 + L_n^4}{L_{n+3}L_n} \geq 2L_{n+3}L_n + 4(L_{n+3} - L_n)^2 = 16L_{n+1}^2 + 4L_{n+1}L_n + 2L_n^2,$$

and

$$2L_{n+4}^2 = 18L_{n+1}^2 + 24L_{n+1}L_n + 8L_n^2.$$

Combining these, we obtain

$$\begin{aligned}\frac{L_n^4 + L_{n+1}^4}{L_nL_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1}L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3}L_n} &\geq 28L_{n+1}^2 + 8L_{n+1}L_n + 10L_n^2\\ &= 2L_{n+4}^2 + 10L_{n+1}^2 - 16L_{n+1}L_n + 2L_n^2\\ &= 2L_{n+4}^2 - 6L_{n+1}L_n + 10L_{n+1}L_{n-1} + 2L_n^2\\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 6L_{n+1}L_{n-2} + 2L_n^2\\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-1} - 4L_{n+1}L_{n-2} + 2L_nL_{n-1} - 2L_{n-1}L_{n-2}\\ &= 2L_{n+4}^2 + 4L_{n+1}L_{n-3} + L_{n-1}^2\\ &> 2L_{n+4}^2.\end{aligned}$$

4063. *Proposed by Marcel Chiriţă.*

Let a, b, c be real numbers greater than or equal to 3. Show that

$$\min\left(\frac{a^2b^2 + 3b^2}{b^2 + 27}, \frac{b^2c^2 + 3c^2}{c^2 + 27}, \frac{a^2c^2 + 3a^2}{a^2 + 27}\right) \leq \frac{abc}{9}.$$

We received six submissions all of which were correct. We present a composite of the similar solutions by Arkady Alt and Leonard Guigiuc.

Suppose to the contrary that

$$\min\left(\frac{a^2b^2+3b^2}{b^2+27},\ \frac{b^2c^2+3c^2}{c^2+27},\ \frac{a^2c^2+3a^2}{a^2+27}\right) > \frac{abc}{9}.$$

Then we have $\prod_{cyc} \frac{a^2b^2+3b^2}{b^2+27} > \frac{a^3b^3c^3}{9^3}$, so $\prod_{cyc} \frac{a^2+3}{a^2+27} > \frac{abc}{9^3}$.

But since $\frac{a}{9} - \frac{a^2+3}{a^2+27} = \frac{a^3-9a^2+27a-27}{9(a^2+27)} = \frac{(a-3)^2}{9(a^2+27)} \geq 0$, we have $\frac{a^2+3}{a^2+27} \leq \frac{a}{9}$.

Similarly, $\frac{b^2+3}{b^2+27} \leq \frac{b}{9}$ and $\frac{c^2+3}{c^2+27} \leq \frac{c}{9}$.

Hence, $\prod_{cyc} \frac{a^2+3}{a^2+27} > \frac{abc}{9^3}$ is a contradiction.

4064. *Proposed by Michel Bataille.*

In the plane of a triangle ABC, let Γ be a circle whose centre O is not on the sidelines AB, BC, CA. Let A', B', C' be the poles of the lines BC, CA, AB with respect to Γ, respectively. Prove that

$$\frac{OA' \cdot B'C'}{OA \cdot BC} = \frac{OB' \cdot C'A'}{OB \cdot CA} = \frac{OC' \cdot A'B'}{OC \cdot AB}.$$

We received five solutions, all correct, and present the solution by Joel Schlosberg, slightly modified by the editor.

One way to define the pole A' of the line BC with respect to the circle Γ is by reciprocation, namely A' is the inverse in Γ of the foot of the perpendicular from O to BC. [See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (The Mathematical Association of America, 1967), Section 6.1.] Conversely, if D is the inverse of A in Γ, then the polar of A, namely $B'C'$, is the line through D that is perpendicular to OA. We shall use three immediate consequences of this definition. If r is the radius of Γ, then $OA \cdot OD = r^2$, or

$$OA = \frac{r^2}{OD}. \tag{1}$$

Since $OB' \perp CA$ and $OC' \perp AB$, $\angle B'OC'$ is equal to or supplementary to $\angle BAC$. Let R be the circumradius of $\triangle ABC$. By the law of sines,

$$BC = 2R \sin \angle BAC = 2R \sin \angle B'OC'. \tag{2}$$

Finally, since each is the area of $\triangle OB'C'$,

$$\frac{1}{2}B'C' \cdot OD = \frac{1}{2}OB' \cdot OC' \sin \angle B'OC'. \tag{3}$$