- if $\{\lambda_1, \lambda_2\} = \{3, 5\}$, then $\chi(x) = (x-3)(x-5)$ and
$$x^3 - 5x^2 + 6x = (x+3)\chi(x) + 15x - 45.$$
Hence $A = 15X - 45$, which gives us
$$X = \frac{1}{15}(A + 45I_2) = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}.$$

The proof is complete.

4036. *Proposed by Arkady Alt.*

Let a, b and c be non-negative real numbers. Prove that for any real $k \geq \frac{11}{24}$ we have:

$$k(ab+bc+ca)(a+b+c) - (a^2c + b^2a + c^2b) \leq \frac{(3k-1)(a+b+c)^3}{9}.$$

We received five submissions all of which are correct. We present the solution by the proposer, slightly modified by the editor.

Due to cyclic symmetry of the functions involved, we may assume that $c = \min\{a, b, c\}$.

Let $x = a - c$ and $y = b - c$. Then $x, y, c \geq 0, a = x + c,\ b = y + c$, and $a + b + c = x + y + 3c$.

The given inequality is equivalent to

$$(3k-1)(a+b+c)^3 - 9k(ab+bc+ca)(a+b+c) + 9\left(a^2c + b^2a + c^2b\right) \geq 0$$

or

$$(3k-1)(x+y+3c)^3 - 9k((x+c)(y+c) + c(x+y+2c))(x+y+3c) \\ + 9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \geq 0. \quad (1)$$

Let $F(x, y, c)$ denote the expression on the left hand side of (1), and set $p = x + y$ and $q = xy$.
Since $9k((x+c)(y+c) + c(x+y+2c))(x+y+3c) = 9k(3c^2 + 2pc + q)(p+3c)$ and

$$\begin{aligned} &9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \\ &= 9(cx^2 + cy^2 + 2cxy + 3c^2(x+y) + xy^2 + 3c^3) \\ &= 9(cp^2 + 3c^2p + 3c^3 + xy^2) \\ &= 9cp^2 + 27c^2p + 27c^3 + 9xy^2 \\ &= (p+3c)^3 - p^3 + 9xy^2, \end{aligned}$$

we have

$$\begin{aligned}F(x,y,c) &= (3k-1)(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) + (p+3c)^3 - p^3 + 9xy^2\\ &= 3k(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) - p^3 + 9xy^2\\ &= 3k(p^3+9cp^2+27c^2p+27c^3) - 9k(2cp^2+9c^2p+9c^3+pq+3cq)\\ &\qquad - p^3 + qxy^2\\ &= (3k-1)p^3 + 9ckp^2 - 27ckq - 9kpq + 9xy^2\\ &= (3k-1)(x+y)^3 + 9ck(x+y)^2 - 27ckxy - 9kxy(x+y) + 9xy^2\\ &= (3k-1)(x^3+y^3+3xy(x+y)) + 9ck(x^2+2xy+y^2) - 27ckxy\\ &\qquad - 9kxy(x+y) + 9xy^2\\ &= (3k-1)x^3 + 6xy^2 - 3x^2y + 9ck(x^2-xy+y^2) + (3k-1)y^3. \qquad (2)\end{aligned}$$

Clearly, $9ck(x^2-xy+y^2)+(3k-1)y^3 \geq 0$. Furthermore,

$$(3k-1)x^3 + 6xy^2 - 3x^2y = x((3k-1)x^2 - 3xy + 6y^2) \geq 0$$

since the discriminant of $(3k-1)x^2 - 3xy + 6y^2$ is

$$9y^2 - 24(3k-1)y^2 = 3(11-24k)y^2 \leq 0$$

and $3k-1>0$. Hence, from (2) we conclude that $F(x,y,c) \geq 0$ which by (1) completes the proof.

4037. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that the area of UVW is independent of the chosen point P on γ.

We received six correct and complete solutions. We present the solution by Oliver Geupel. Ricard Peiró and Prithwijt De submitted similar solutions.

We prove that triangle UVW is similar to triangle ABC. As a consequence, since γ is the circumcircle of UVW, the area of triangle UVW is

$$[UVW] = \frac{r^2}{R^2}[ABC],$$

where r and R denote the inradius and the circumradius, respectively, of $\triangle ABC$.

Let $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{U}$, $\hat{V}$, and $\hat{W}$ denote measures of the the interior angles of the triangles ABC and UVW. Since $PV \perp AC$ and $PW \perp AB$, the size of $\angle VPW$ is $180^\circ - \hat{A}$. Also, since the points P, U, V, and W are concyclic, $\angle VUW$ is equal to either $\angle VPW$ or $180^\circ - \angle VPW$. Hence, $\hat{U} \in \{\hat{A},\ 180^\circ - \hat{A}\}$. Analogously, $\hat{V} \in \{\hat{B},\ 180^\circ - \hat{B}\}$ and $\hat{W} \in \{\hat{C},\ 180^\circ - \hat{C}\}$.

We show that $(\hat{U}, \hat{V}, \hat{W}) = (\hat{A}, \hat{B}, \hat{C})$.