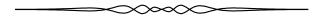
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(4), p. 169-172.



4031. Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.

Prove that

$$\frac{2F_1^4 + F_2^4 + F_3^4}{F_1^2 + F_3^2} + \frac{2F_2^4 + F_3^4 + F_4^4}{F_2^2 + F_4^2} + \dots + \frac{2F_n^4 + F_1^4 + F_2^4}{F_n^2 + F_2^2} > 2F_n F_{n+1},$$

where F_n represents the *n*th Fibonacci number $(F_0 = 0, F_1 = 1 \text{ and } F_{n+2} = F_n + F_{n+1} \text{ for all } n \geq 1)$.

We received five correct solutions. We present two solutions.

Editor's comments. When n=1, the interpretation of the left side is not clear, while when n=2, we obtain equality. Therefore, we suppose that $n\geq 3$.

Solution 1, by Adnan Ali and the proposers (independently).

Observe that, for positive x, y, z,

$$\frac{2x^2 + y^2 + z^2}{x + z} \ge x + y$$

with equality if and only if x = y = z, since this inequality is equivalent to $(x-y)^2 + (y-z)^2 + (z-x)^2 \ge 0$. It follows that the left side of the inequality is greater than

$$(F_1^2 + F_2^2) + (F_2^2 + F_3^2) + \dots + (F_{n-1}^2 + F_n^2) + (F_n^2 + F_1^2) = 2\sum_{k=1}^n F_k^2.$$

Since $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ (easily obtained by induction for $n \geq 1$), the result follows.

Solution 2, by Arkady Alt.

For positive x, y, z,

$$\frac{x^2}{y+z} \ge \frac{4x-y-z}{4}$$

with equality iff 2x = y + z. This implies that

$$\frac{2F_i^4 + F_j^4 + F_k^4}{F_i^2 + F_k^2} > \frac{2(4F_i^2 - F_i^2 - F_k^2) + (4F_j^2 - F_i^2 - F_k^2) + (4F_k^2 - F_i^2 - F_k^2)}{4}$$

$$= F_i^2 + F_i^2,$$

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for distinct i, j, k (since not all of F_i, F_j, F_k are equal to $F_i + F_k$). Thus the left side is greater than $2\sum_{k=1}^n F_k^2 = 2F_nF_{n+1}$.

4032. Proposed by Dan Stefan Marinescu and Leonard Giugiuc.

Prove that in any triangle ABC with sides a, b and c, inradius r and exradii r_a, r_b, r_c , we have:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \ge 2\sqrt{3r(r_a + r_b + r_c)}$$
.

We received 13 correct solutions. We present two solutions.

Solution 1, by Titu Zvonaru.

Using Ravi's substitutions (a = y + z, b = z + x, c = x + y, with x, y, z > 0), we have

$$[ABC] = \sqrt{xyz(x+y+z)}, \ r = \sqrt{\frac{xyz}{x+y+z}}, \ r_a = \frac{\sqrt{xyz(x+y+z)}}{x},$$

so that

$$r(r_a + r_b + r_c) = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = xy + yz + zx.$$

We have to prove that

$$\sqrt{(x+y)(y+z)} + \sqrt{(y+z)(z+x)} + \sqrt{(z+x)(x+y)} \ge 2\sqrt{3(xy+yz+zx)}.$$

Using Minkowski's inequality and the inequality $(x + y + z)^2 \ge 3(xy + yz + zx)$, we obtain

$$\begin{split} &\sqrt{(x+y)(y+z)} + \sqrt{(y+z)(z+x)} + \sqrt{(z+x)(x+y)} \\ &= \sqrt{x^2 + (xy + yz + zx)} + \sqrt{y^2 + (xy + yz + zx)} + \sqrt{z^2 + (xy + yz + zx)} \\ &\geq \sqrt{(x+y+z)^2 + (3\sqrt{xy + yz + zx})^2} \\ &\geq \sqrt{3(xy + yz + zx)} + 9(xy + yz + zx) \\ &= 2\sqrt{3(xy + yz + zx)}. \end{split}$$

Equality holds if and only if x=y=z; that is, if and only if triangle ABC is equilateral.

Solution 2, composite of similar solutions by Sĕfket Arslanagic and Kee-Wai Lau.

By the known equality

$$r_a + r_b + r_c = 4R + r,$$

the given inequality is equivalent to

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \ge 2\sqrt{3r(4R+r)}$$
.