# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2015: 41(4), p. 169–172.*

**4031**. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Prove that

$$\frac{2F_1^4+F_2^4+F_3^4}{F_1^2+F_3^2}+\frac{2F_2^4+F_3^4+F_4^4}{F_2^2+F_4^2}+\cdots+\frac{2F_n^4+F_1^4+F_2^4}{F_n^2+F_2^2}>2F_nF_{n+1},$$

where $F_n$ represents the $n$th Fibonacci number ($F_0=0, F_1=1$ and $F_{n+2}=F_n+F_{n+1}$ for all $n\geq 1$).

*We received five correct solutions. We present two solutions.*

*Editor's comments.* When $n=1$, the interpretation of the left side is not clear, while when $n=2$, we obtain equality. Therefore, we suppose that $n\geq 3$.

*Solution 1, by Adnan Ali and the proposers (independently).*

Observe that, for positive $x,y,z$,

$$\frac{2x^2+y^2+z^2}{x+z}\geq x+y$$

with equality if and only if $x=y=z$, since this inequality is equivalent to $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$. It follows that the left side of the inequality is greater than

$$(F_1^2+F_2^2)+(F_2^2+F_3^2)+\cdots+(F_{n-1}^2+F_n^2)+(F_n^2+F_1^2)=2\sum_{k=1}^{n}F_k^2.$$

Since $\sum_{k=1}^n F_k^2=F_nF_{n+1}$ (easily obtained by induction for $n\geq 1$), the result follows.

*Solution 2, by Arkady Alt.*

For positive $x,y,z$,

$$\frac{x^2}{y+z}\geq\frac{4x-y-z}{4}$$

with equality iff $2x=y+z$. This implies that

$$\begin{aligned}\frac{2F_i^4+F_j^4+F_k^4}{F_i^2+F_k^2}&>\frac{2(4F_i^2-F_i^2-F_k^2)+(4F_j^2-F_i^2-F_k^2)+(4F_k^2-F_i^2-F_k^2)}{4}\\&=F_i^2+F_j^2,\end{aligned}$$

for distinct $i, j, k$ (since not all of $F_i$, $F_j$, $F_k$ are equal to $F_i + F_k$). Thus the left side is greater than $2\sum_{k=1}^{n} F_k^2 = 2F_nF_{n+1}$.

**4032**. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Prove that in any triangle $ABC$ with sides $a, b$ and $c$, inradius $r$ and exradii $r_a, r_b, r_c$, we have:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(r_a + r_b + r_c)}.$$

*We received 13 correct solutions. We present two solutions.*

*Solution 1, by Titu Zvonaru.*

Using Ravi's substitutions ($a = y + z$, $b = z + x$, $c = x + y$, with $x, y, z > 0$), we have

$$[ABC] = \sqrt{xyz(x + y + z)},\ r = \sqrt{\frac{xyz}{x + y + z}},\ r_a = \frac{\sqrt{xyz(x + y + z)}}{x},$$

so that

$$r(r_a + r_b + r_c) = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = xy + yz + zx.$$

We have to prove that

$$\sqrt{(x + y)(y + z)} + \sqrt{(y + z)(z + x)} + \sqrt{(z + x)(x + y)} \geq 2\sqrt{3(xy + yz + zx)}.$$

Using Minkowski's inequality and the inequality $(x + y + z)^2 \geq 3(xy + yz + zx)$, we obtain

$$\begin{aligned}
&\sqrt{(x + y)(y + z)} + \sqrt{(y + z)(z + x)} + \sqrt{(z + x)(x + y)}\\
&= \sqrt{x^2 + (xy + yz + zx)} + \sqrt{y^2 + (xy + yz + zx)} + \sqrt{z^2 + (xy + yz + zx)}\\
&\geq \sqrt{(x + y + z)^2 + (3\sqrt{xy + yz + zx})^2}\\
&\geq \sqrt{3(xy + yz + zx) + 9(xy + yz + zx)}\\
&= 2\sqrt{3(xy + yz + zx)}.
\end{aligned}$$

Equality holds if and only if $x = y = z$; that is, if and only if triangle $ABC$ is equilateral.

*Solution 2, composite of similar solutions by Šefket Arslanagic and Kee-Wai Lau.*

By the known equality

$$r_a + r_b + r_c = 4R + r,$$

the given inequality is equivalent to

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(4R + r)}.$$

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