



Simultaneous Transformations of Möbius Functions and their applications for solving recurrences

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Dedicated to Themistocles M. Rassias on the occasion of his 75th Birthday

INTRODUCTION

One of the ways to equivalently transform recurrence $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$ is simultaneous transformation, namely if $h : \mathbb{R} \rightarrow \mathbb{R}$ is one-to-one correspondence, then the substitution $x_n = h(t_n)$, $n \in \mathbb{N}$ leads to

$$h(t_{n+1}) = f(h(t_n)) \iff t_{n+1} = (h^{-1} \circ f \circ h)(t_n).$$

In a very particular case, we may take $h(t) := kt$ or $h(t) := kt + l$, where k and l are some real numbers. It is convenient to represent a simultaneous transformation by applying it to an equation of the form $y = f(x)$. Using the substitution $x := h(u)$ and $y := h(v)$ we can transform the equation $y = f(x)$ into $v = h^{-1} \circ f \circ h(u)$ and then attempt to find such a function h , chosen from a given class of functions, which yields a more convenient form of the transformed equation.

In this article we will consider applications of this idea to recurrence $x_{n+1} = f(x_n)$, where f be a Möbius function (fraction-linear function), namely $f(x) = \frac{ax+b}{cx+d}$ with goal to find for recurrence

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}$$

simplest equivalent representation convenient for finding closed-form solution. Let $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be matrix associated with $f(x) = \frac{ax+b}{cx+d}$, which we call matrix of the f . It is easy to see that the matrix H corresponding to

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the composition $h = f \circ g$ of two Moebius functions f and g is equal to the product of their matrices F and G , that is $H = F \cdot G$. Since a linear function $l(x) = kx + l$ can be regarded as a special case of a Möbius function, namely $l(x) = \frac{kx + l}{0x + 1}$, its matrix representation is $L = \begin{pmatrix} k & l \\ 0 & 1 \end{pmatrix}$.

1. Transformation $y = \frac{ax + b}{cx + d}$ to the form. $v = p - \frac{q}{u}$.

Starting with $y = \frac{ax + b}{cx + d}$ we have

$$cy + d = c \cdot \frac{ax + b}{cx + d} + d = \frac{(a + d)(cx + d) - (ad - bc)}{cx + d}$$

Now, denoting $v := cy + d$ and $u := cx + d$, we obtain

$v = \frac{(a + d)v - (ad - bc)}{v}$ and moreover $x = \frac{u}{c} - \frac{d}{c}$, $y = \frac{v}{c} - \frac{d}{c}$. Hence, by using the simultaneous linear transformation $x = l(u)$ and $y = l(v)$, where $l(t) := \frac{1}{c} \cdot t + \left(-\frac{d}{c}\right)$, we can reduce equation $y = \frac{ax + b}{cx + d}$ to the special form $v = p - \frac{q}{u}$, where $p = a + d$, $q = ad - bc$. In the matrix form

$$\begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix} = L^{-1}FL, \text{ where } p = a + d, q = ad - bc,$$

$$L = \begin{pmatrix} \frac{1}{c} & -\frac{d}{c} \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } L^{-1} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$

APPLICATION

For $f(u) = p - \frac{q}{u}$ the recurrence $u_{n+1} = f(u_n)$

becomes $u_{n+1} = p - \frac{q}{u_n}$, $n \in \mathbb{N}$. Then denoting $t_n := \prod_{k=1}^n u_k$, $n \in \mathbb{N}$ and

$t_0 := 1$ we obtain $u_n = \frac{t_n}{t_{n-1}}$, $n \in \mathbb{N}$ and, therefore,

$$u_{n+1} = p - \frac{q}{u_n} \iff \frac{t_{n+1}}{t_n} = p - q \cdot \frac{t_{n-1}}{t_n} \iff$$

$$t_{n+1} - pt_n + qt_{n-1} = 0, n \in \mathbb{N}, \text{ where } t_0 = 1, t_1 = u_1. \quad (1)$$

Here possible 3 cases:

a) If $p^2 > 4q$ then characteristic quadratic equation have two distinct solution k_1, k_2 and we have $t_n = c_1 k_1^n + c_2 k_2^n, n \in \mathbb{N}$. Thus,

$$u_n = \frac{c_1 k_1^{n+1} + c_2 k_2^{n+1}}{c_1 k_1^n + c_2 k_2^n}, n \in \mathbb{N}.$$

Constants c_1, c_2 can be obtained from $t_0 = 1 \iff c_1 + c_2 = 1$ and $c_1 k_1 + c_2 k_2 = t_1 = x_1$.

b) If $p^2 = 4q$ then $u_{n+1} = p - \frac{p^2}{4u_n}, n \in \mathbb{N}$ and after replacing p with $2p$ this recurrence becomes

$$\begin{aligned} u_{n+1} = 2p - \frac{p^2}{u_n} &\iff u_{n+1} - p = \frac{p(u_n - p)}{u_n} \iff (u_{n+1}/p) - 1 = \\ &= \frac{(u_n/p) - 1}{(u_n/p)} \iff t_{n+1} = \frac{t_n}{t_n + 1}, \text{ where } t_n := (u_n/p) - 1. \end{aligned}$$

Since

$$\begin{aligned} t_{n+1} = \frac{t_n}{t_n + 1} &\iff \frac{1}{t_{n+1}} = \frac{1}{t_n} + 1, n \in \mathbb{N} \iff \frac{1}{t_n} = \\ &= n + c \iff t_n = \frac{1}{n + c}, n \in \mathbb{N} \end{aligned}$$

then $\frac{u_n}{p} - 1 = \frac{1}{n + c} \iff u_n = p + \frac{p}{n + c}$, where constant c can be obtained from $u_1 = p + \frac{p}{1 + c}$.

c) If $p^2 < 4q$ then characteristic quadratic equation for recurrence **(1)** have no real solution. In this case denoting $\varphi := \arccos \frac{p}{2\sqrt{q}}$ we obtain

$$p = 2\sqrt{q} \cos \varphi.$$

Then recurrence **(1)** becomes

$$\begin{aligned} t_{n+1} - 2\sqrt{q} \cos \varphi t_n + q t_{n-1} &= 0 \iff \\ \iff \frac{t_{n+1}}{(\sqrt{q})^{n+1}} - 2 \cos \varphi \cdot \frac{t_n}{(\sqrt{q})^n} + \frac{t_{n-1}}{(\sqrt{q})^{n-1}} &= 0, \end{aligned}$$

$$n \in \mathbb{N} \iff s_{n+1} - 2 \cos \varphi \cdot s_n + s_{n-1} = 0$$

where

$$s_n := \frac{t_n}{(\sqrt{q})^n}, n \in \mathbb{N} \cup \{0\}.$$

Since sequences $(\cos n\varphi), (\sin n\varphi)$ both be linearly independent solutions of the recurrence $s_{n+1} - 2 \cos \varphi \cdot s_n + s_{n-1} = 0$ then any solution s_n of this recurrence can be represented as linear combination of $(\cos n\varphi)$ and $(\sin n\varphi)$, that is

$$\begin{aligned} s_n &= c_1 \cos n\varphi + c_2 \sin n\varphi \iff \frac{t_n}{(\sqrt{q})^n} = c_1 \cos n\varphi + c_2 \sin n\varphi \iff \\ t_n &= (\sqrt{q})^n (c_1 \cos n\varphi + c_2 \sin n\varphi), n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Therefore,

$$u_n = \sqrt{q} \cdot \frac{c_1 \cos (n+1)\varphi + c_2 \sin (n+1)\varphi}{c_1 \cos n\varphi + c_2 \sin n\varphi}, n \in \mathbb{N},$$

where values of constants c_1, c_2 can be determined from initial conditions u_0, u_1 .

2. Transformation $y = \frac{ax+b}{cx+d}$ to the form $v = pu + q$.

Let $f(x) = \frac{ax+b}{cx+d}$. For certain values of the parameters a, b, c, d , there exists a simultaneous Möbius transformation that reduces the equation $y = f(x)$ to a linear one. We will look for such a function of the form $h(x) = \frac{tx+1}{x}$, where t is an undetermined parameter to be chosen so that the transformed function satisfies

$$h(v) = f(h(u)) \iff v = h^{-1} \circ f \circ h(u) \iff v = pu + q$$

for some numbers p, q .

The matrix corresponding to h is $H = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}$. Then $H^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix}$ and

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} c & d \\ a - ct & b - dt \end{pmatrix} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} ct + d & c \\ at - ct^2 + b - dt & a - ct \end{pmatrix} \end{aligned}$$

We require that the lower-left entry be zero to make the transformation linear; hence:

$$at - ct^2 + b - dt = 0 \iff ct^2 - t(a - d) - b = 0.$$

The discriminant is

$$D = (a - d)^2 + 4bc = (a + d)^2 - 4(ad - bc).$$

Thus, a real t exists whenever $D \geq 0$. For such a t we obtain

$$H^{-1}FH = \begin{pmatrix} ct + d & c \\ 0 & a - ct \end{pmatrix}$$

which corresponds to the linear function $v = \frac{ct + d}{a - ct}u + \frac{c}{a - ct}$.

Since

$$\text{Dom}(h) = \mathbb{R} \setminus \{0\}, \text{Dom}(h^{-1}) = \mathbb{R} \setminus \{t\}$$

and

$$\text{Dom}(f) = \mathbb{R} \setminus \left\{-\frac{d}{c}\right\}, \text{Dom}(f^{-1}) = \mathbb{R} \setminus \left\{\frac{a}{c}\right\}$$

then equivalency

$$y = f(x) \iff v = h^{-1} \circ f \circ h(u)$$

holds for $y \notin \left\{\frac{a}{c}, t\right\}$, $x \notin \left\{-\frac{d}{c}, v\right\}$, $u, v \neq 0$.

Finally, the system

$$\begin{cases} ct^2 - t(a - d) - b = 0 \\ a - ct = 0 \end{cases}$$

has no solution because $ad - bc \neq 0$. Therefore $a - ct \neq 0$ for t satisfying to equation

$$ct^2 - t(a - d) - b = 0.$$

Thus, the Möbius transformation $y = \frac{ax + b}{cx + d}$ can be reduced to a linear form $v = pu + q$ if and only if

$$(a + d)^2 \geq 4(ad - bc).$$

3. Transformation $y = \frac{ax+b}{cx+d}$ **to the form** $v = \frac{pu-q}{p+uq}$.

When $D = (a+d)^2 - 4(ad-bc) < 0 \implies c \neq 0$ the situation becomes more complicated. In this case, there is no simultaneous Möbius transformation that can reduce the equation $y = f(x)$ to a linear form. However, it is still possible to achieve a simplification by using a composition of a linear function and the tangent function. Indeed, let us first attempt to use a linear function $l(x) = tx + s$ to reduce equation $y = f(x)$ to the form $v = \frac{pu-q}{p+uq}$, where p and q are real parameters to be determined.

By applying the simultaneous linear transformation $y = l(v)$, $x = l(u)$, where $l(\theta) = t\theta + s$ to the equation $y = f(x) = \frac{ax+b}{cx+d}$, we obtain

$$tv + s = \frac{a(tu + s) + b}{c(tu + s) + d} \iff v = \frac{u(at - sct) + b + as - cs^2 - ds}{t^2cu + (cs + d)t}$$

To simplify this expression into the form $v = \frac{pu-q}{p+uq}$ we must ensure that the coefficients satisfy the system

$$\begin{cases} at - sct = (cs + d)t \\ b + as - cs^2 - ds = -ct^2 \end{cases} \iff \begin{cases} 2cs = a - d \\ \frac{cs^2 + ds - as - b}{c} = t^2 \end{cases} \iff \begin{cases} s = \frac{a-d}{2c} \\ t^2 = \frac{cs^2 - s(a-d) - b}{c} \end{cases} \iff \begin{cases} s = \frac{a-d}{2c} \\ t^2 = -\frac{(a-d)^2 + 4bc}{4c^2} = -\frac{D}{4c^2} \end{cases}$$

Hence, for $s = \frac{a-d}{2c}$ and $t = \frac{\sqrt{-D}}{2c}$ we obtain

$$\begin{aligned} v &= \frac{u \cdot \frac{t(a+d)}{2} - ct^2}{t^2cu + \frac{a+d}{2}t} = \frac{u \cdot \frac{a+d}{2} - \frac{\sqrt{-D}}{2}}{\frac{\sqrt{-D}}{2}u + \frac{a+d}{2}} = \\ &= \frac{u \cdot (a+d) - \sqrt{-D}}{\sqrt{-D} \cdot u + (a+d)} = \frac{u \cdot \frac{a+d}{\sqrt{-D}} - 1}{u + \frac{a+d}{\sqrt{-D}}} \end{aligned}$$

So, $v = \frac{u \cdot \frac{a+d}{\sqrt{-D}} - 1}{u + \frac{a+d}{\sqrt{-D}}}$. Now, let

$$v := \cot^{-1}(v), \varphi := \cot^{-1}(u) \text{ and } \alpha := \cot^{-1}\left(\frac{a+d}{\sqrt{-D}}\right)$$

Then

$$\cot v = \frac{\cot \varphi \cdot \frac{a+d}{\sqrt{-D}} - 1}{\cot \varphi + \frac{a+d}{\sqrt{-D}}} = \cot(\varphi + \alpha),$$

which implies $v = \varphi + \alpha$, (if $\varphi + \alpha \in (0, \pi)$).

4. Application for Moebius recurrence.

Consider the Möbius recurrence

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \text{ with } ad - bc \neq 0.$$

We distinguish two cases according to the discriminant

$$D = (a+d)^2 - 4(ad-bc)$$

Case 1. $D \geq 0$.

Then there exists a real t satisfying $ct^2 - t(a-d) - b = 0$ and for such t let

$$u_n := \frac{1}{x_n - t}$$

Then by substituting $x_n = t + \frac{1}{u_n}$ into the recurrence

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}$$

we obtain linear recurrence

$$u_{n+1} = \frac{ct+d}{a-ct}u_n + \frac{c}{a-ct}, n \in \mathbb{N} \cup \{0\}$$

Case 2. $D < 0$.

Let $s := \frac{a-d}{2c}$, $t := \frac{\sqrt{-D}}{2c}$ and define $x_n := \frac{u_n \sqrt{-D} + a - d}{2c}$. Then substitution yields the recurrence

$$u_{n+1} = \frac{u_n \cdot \frac{a+d}{\sqrt{-D}} - 1}{u_n + \frac{a+d}{\sqrt{-D}}}$$

Let $\alpha := \cot^{-1} \left(\frac{a+d}{\sqrt{-D}} \right)$, $\varphi_0 := \cot^{-1}(u_0)$. Then $u_n = \cot(n\varphi + \alpha)$ and hence

$$x_n = \frac{\sqrt{-D} \cot(n\alpha + \varphi_0) + a - d}{2c}, n \in \mathbb{N} \cup \{0\}.$$

REFERENCE

[1] Octagon Mathematical Magazine (1993-2025)

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