



Rationalization of Fundamental Geometric Inequality and its application

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Dedicated to Themistocles M. Rassias on the occasion of his 75th Birthday

Fundamental Geometric Inequality (Blundon's inequality)

$$\begin{aligned} 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} &\leq \\ \leq s^2 &\leq 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} \end{aligned} \quad (FGI)$$

is a necessary and sufficient condition for three positive real numbers s, R and r to be, respectively, a semiperimeter, a circumradius and an inradius of a triangle. But application of this inequality often becomes difficult due to the presence of quadratic irrationality, and so often proofs of geometric inequalities related to triangle instead often make use of Gerretsen's inequality

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \quad (\text{Gr})$$

However, the use of inequality (Gr) sometimes gives undesired results, since it only provides a necessary condition. For example, this becomes clearly visible when considering an attempt to solve the following problem [ref 1] with an application of Gerretsen's inequality:

Let a, b, c be side-lengths of a triangle and let S be its area. Let R and r be the circumradius and the inradius of the triangle, respectively. Prove that

$$\cot^2 A + \cot^2 B + \cot^2 C \geq \frac{1}{5} \left(31 - 52 \cdot \frac{r}{R} \right)$$

An attempt to solve this problem with Gerretsen's inequality looks as below:

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Since*

$$\sum_{cyc} \cot^2 A = \frac{(s^2 - r^2 - 4Rr)^2}{4s^2r^2} - 2$$

and

$$\frac{(s^2 - r^2 - 4Rr)^2}{4s^2r^2}$$

is increasing in s because

$$\frac{(s^2 - r^2 - 4Rr)^2}{4s^2r^2} = \frac{\left(s - \frac{r^2 + 4Rr}{s}\right)^2}{4r^2}$$

then using Gerretsen's inequality $16Rr - 5r^2 \leq s^2$ we obtain

$$\frac{(s^2 - r^2 - 4Rr)^2}{4s^2r^2} \geq \frac{(16Rr - 5r^2 - r^2 - 4Rr)^2}{4(16Rr - 5r^2)r^2} = \frac{9(2R - r)^2}{r(16R - 5r)}$$

Thus it seem as if the expression $\frac{9(2R - r)^2}{r(16R - 5r)} - 2$ can potentially be used for completing the proof as the rational R, r -minorant for $\sum_{cyc} \cot^2 A$. But

unfortunately, the difference

$$\frac{9(2R - r)^2}{r(16R - 5r)} - 2 - \frac{1}{5} \left(31 - 52 \cdot \frac{r}{R}\right) = \frac{2(R - 2r)(90R^2 - 238Rr + 65r^2)}{5Rr(16R - 5r)}$$

isn't is not positive for any $R \geq 2r$ (Euler's inequality). Since

$$90 \cdot (2r)^2 - 238(2r) \cdot r + 65r^2 = -51r^2 < 0$$

then for $R > 2r$ and close enough to $2r$, $90R^2 - 238Rr + 65r^2$ will be less than 0, and therefore for such R

$$\frac{2(R - 2r)(90R^2 - 238Rr + 65r^2)}{5Rr(16R - 5r)} < 0$$

Hence the chosen minorant is not good enough.

Remark 1. Using $\frac{9(2R - r)^2}{r(16R - 5r)} - 2$ as the rational R, r -minorant for $\sum_{cyc} \cot^2 A$ we can prove inequality (2)

$$\cot^2 A + \cot^2 B + \cot^2 C \geq \frac{1}{9} \left(49 - 80 \frac{r}{R} \right) \quad (2)$$

which is weaker than inequality of the problem because

$$\frac{1}{5} \left(31 - 52 \cdot \frac{r}{R} \right) - \frac{1}{9} \left(49 - 80 \frac{r}{R} \right) = \frac{34(R-2r)}{45R} \geq 0$$

Indeed,

$$\frac{9}{r} \frac{(2R-r)^2}{16R-5r} - 2 - \frac{1}{9} \left(49 - 80 \frac{r}{R} \right) = \frac{4(81R-25r)(R-2r)^2}{9Rr(16R-5r)} \geq 0$$

Remark 2. Even using inequality [ref 2]

$$s^2 \geq 16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} = \frac{r(16R^2 - 20Rr + 3r^2)}{R-r}$$

can't help in proof of inequality of the problem. Indeed,

$$\begin{aligned} \sum_{cyc} \cot^2 A &= \frac{(s^2 - r^2 - 4Rr)^2}{4s^2r^2} - 2 \geq \\ &\geq \frac{\left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} - r^2 - 4Rr \right)^2}{4 \left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \right) r^2} - 2 = \\ &= \frac{(12R^2 - 17Rr + 4r^2)^2}{4r(R-r)(16R^2 - 20Rr + 3r^2)} - 2 \end{aligned}$$

But unfortunately, the difference

$$\begin{aligned} &\frac{(12R^2 - 17Rr + 4r^2)^2}{4r(R-r)(16R^2 - 20Rr + 3r^2)} - 2 - \frac{1}{5} \left(31 - 52 \cdot \frac{r}{R} \right) = \\ &= \frac{1}{20} (R-2r) \frac{720R^4 + 4709R^2r^2 - 2522Rr^3 - 3224R^3r + 312r^4}{Rr(R-r)(16R^2 - 20Rr + 3r^2)} \end{aligned}$$

isn't is not positive for any $R \geq 2r$ (Euler's inequality). Since

$$720 \cdot (2r)^4 + 4709(2r)^2r^2 - 2522(2r)r^3 - 3224(2r)^3r + 312r^4 = -168r^4$$

is negative then for $R > 2r$ and close enough to $2r$,

$$720R^4 + 4709R^2r^2 - 2522Rr^3 - 3224R^3r + 312r^4$$

will be less than 0, and therefore for such R

$$\frac{1}{20} (R - 2r) \frac{720R^4 + 4709R^2r^2 - 2522Rr^3 - 3224R^3r + 312r^4}{Rr(R - r)(16R^2 - 20Rr + 3r^2)} < 0$$

That is inequality

$$16Rr - 5r^2 + \frac{r^2(R - 2r)}{R - r} \leq s^2$$

which is sharper than $s^2 \leq 4R^2 + 4Rr + r^2$ is also not good enough.

First rationalization of (FGI)

Coming back to (FGI) we denote $\lambda := \sqrt{1 - \frac{2r}{R}}$. Then

$$\lambda \in [0, 1), \quad r = \frac{(1 - \lambda^2)R}{2}$$

$R - 2r = \lambda^2 R$ and

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} =$$

$$2R^2 + 10R \cdot \frac{(1 - \lambda^2)R}{2} - \left(\frac{(1 - \lambda^2)R}{2} \right)^2 - 2R^2\lambda^3 = \frac{R^2(1 - \lambda)(\lambda + 3)^3}{4}$$

$$2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)} =$$

$$2R^2 + 10R \cdot \frac{(1 - \lambda^2)R}{2} - \left(\frac{(1 - \lambda^2)R}{2} \right)^2 + 2R^2\lambda^3 = \frac{R^2(\lambda + 1)(3 - \lambda)^3}{4}$$

and we obtain

$$\frac{R^2(1 - \lambda)(\lambda + 3)^3}{4} \leq s^2 \leq \frac{R^2(\lambda + 1)(3 - \lambda)^3}{4} \quad (\text{RFGI})$$

(Rationalized FGI).

Using inequality

$$\frac{R^2 (1 - \lambda) (\lambda + 3)^3}{4} \leq s^2$$

we can complete the solution of the problem. We have

$$\begin{aligned} \sum_{cyc} \cot^2 A &= \frac{(s^2 - r^2 - 4Rr)^2}{4s^2 r^2} = \\ &= \frac{\left(s^2 - \left(\frac{(1 - \lambda^2) R}{2} \right)^2 - 4R \frac{(1 - \lambda^2) R}{2} \right)^2}{4s^2 \left(\frac{(1 - \lambda^2) R}{2} \right)^2} = \\ &= \frac{\left(s^2 - \frac{R^2 (1 - \lambda^2) (9 - \lambda^2)}{4} \right)^2}{s^2 (1 - \lambda^2)^2 R^2} \geq \\ &\geq \frac{\left(\frac{R^2 (1 - \lambda) (\lambda + 3)^3}{4} - \frac{R^2 (1 - \lambda^2) (9 - \lambda^2)}{4} \right)^2}{\frac{R^2 (1 - \lambda) (\lambda + 3)^3}{4} (1 - \lambda^2)^2 R^2} = \\ &= \frac{(\lambda^2 + 2\lambda + 3)^2}{(1 - \lambda) (\lambda + 3) (\lambda + 1)^2} \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{(s^2 - r^2 - 4Rr)^2}{4s^2 r^2} - 2 - \frac{1}{5} \left(31 - 52 \cdot \frac{r}{R} \right) \geq \\ &\geq \frac{(\lambda^2 + 2\lambda + 3)^2}{(1 - \lambda) (\lambda + 3) (\lambda + 1)^2} - 2 - \frac{1}{5} (31 - 26 \cdot (1 - \lambda^2)) = \\ &= \frac{2\lambda^2 (13\lambda^4 + 52\lambda^3 + 36\lambda^2 - 12\lambda + 1)}{5 (\lambda + 3) (1 - \lambda) (\lambda + 1)^2} \geq 0 \end{aligned}$$

because

$$13\lambda^4 + 52\lambda^3 + 36\lambda^2 - 12\lambda + 1 = 13\lambda^4 + 52\lambda^3 + (6\lambda - 1)^2 > 0$$

Second rationalization of (FGI).

Let $\rho := \sqrt{R(R-2r)}$. Then $0 \leq \rho < R$, $r = \frac{1}{2} \left(R - \frac{\rho^2}{R} \right)$, $R - 2r = \frac{\rho^2}{R}$ and

$$s^2 \geq 2R^2 + 10R \cdot \frac{1}{2} \left(R - \frac{\rho^2}{R} \right) - \frac{1}{4} \left(R - \frac{\rho^2}{R} \right)^2 - 2 \cdot \frac{\rho^2}{R} \cdot \rho = \frac{(R-\rho)(3R+\rho)^3}{4R^2}$$

$$s^2 \leq 2R^2 + 10R \cdot \frac{1}{2} \left(R - \frac{\rho^2}{R} \right) - \frac{1}{4} \left(R - \frac{\rho^2}{R} \right)^2 + 2 \cdot \frac{\rho^2}{R} \cdot \rho = \frac{(R+\rho)(3R-\rho)^3}{4R^2}$$

So, if instead r we will use $\rho = \sqrt{R(R-2r)}$ then (FGI) becomes rational, namely

$$\frac{(R-\rho)(3R+\rho)^3}{4R^2} \leq s^2 \leq \frac{(R+\rho)(3R-\rho)^3}{4R^2} \quad (RFGI2)$$

As one more application of (RFGI) we will prove the following inequality:

$$s^2 \leq \frac{(2R+r)(10R^2+5Rr+4r^2)}{5R} \quad (3)$$

using (RFGI) because unfortunately, Gerretsen's inequality

$$s^2 \leq 4R^2 + 4Rr + r^2$$

can't help to prove it.

Proof of (3). Since $r = \frac{(1-\lambda^2)R}{2}$ then

$$\begin{aligned} s^2 &\leq \frac{(2R+r)(10R^2+4r^2+5Rr)}{5R} = \\ &= \frac{\left(2R + \frac{(1-\lambda^2)R}{2}\right) \left(10R^2 + 4\left(\frac{(1-\lambda^2)R}{2}\right)^2 + 5R \cdot \frac{(1-\lambda^2)R}{2}\right)}{5R} = \\ &= \frac{R^2(5-\lambda^2)(2\lambda^4-9\lambda^2+27)}{20} \end{aligned}$$

and using $s^2 \leq \frac{R^2 (\lambda + 1) (3 - \lambda)^3}{4}$ **(RFGI)** we obtain

$$\begin{aligned} & \frac{R^2 (5 - \lambda^2) (2\lambda^4 - 9\lambda^2 + 27)}{20} - s^2 \geq \\ & \geq \frac{R^2 (5 - \lambda^2) (2\lambda^4 - 9\lambda^2 + 27)}{20} - \frac{R^2 (\lambda + 1) (3 - \lambda)^3}{4} = \\ & = \frac{1}{10} R^2 \lambda^2 (9 - 2\lambda - \lambda^2) (\lambda - 1)^2 \geq 0 \end{aligned}$$

Or, using right hand side of (RFGI2), namely inequality

$$\begin{aligned} s^2 & \leq \frac{(R + \rho) (3R - \rho)^3}{4R^2} : \text{Since } r = \frac{R^2 - \rho^2}{2R} \text{ then} \\ & \frac{(2R + r) (10R^2 + 4r^2 + 5Rr)}{5R} = \\ & = \frac{\left(2R + \frac{R^2 - \rho^2}{2R}\right) \left(10R^2 + 4\left(\frac{R^2 - \rho^2}{2R}\right)^2 + 5R \cdot \frac{R^2 - \rho^2}{2R}\right)}{5R} = \\ & = \frac{(5R^2 - \rho^2) (27R^4 + 2\rho^4 - 9R^2 \rho^2)}{20R^4} \end{aligned}$$

and, we obtain

$$\begin{aligned} & \frac{(5R^2 - \rho^2) (27R^4 + 2\rho^4 - 9R^2 \rho^2)}{20R^4} - s^2 \geq \\ & \geq \frac{(5R^2 - \rho^2) (27R^4 + 2\rho^4 - 9R^2 \rho^2)}{20R^4} - \frac{(R + \rho) (3R - \rho)^3}{4R^2} = \\ & = \frac{\rho^2 (9R^2 - \rho^2 - 2R\rho) (R - \rho)^2}{10R^4} \geq 0 \end{aligned}$$

Remark. Inequality $s^2 \leq \frac{(2R + r) (10R^2 + 5Rr + 4r^2)}{5R}$ is sharper than

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

because

$$4R^2 + 4Rr + 3r^2 - \frac{(2R + r) (10R^2 + 5Rr + 4r^2)}{5R} = \frac{2r^2 (R - 2r)}{5R} \geq 0$$

* We have

$$\begin{aligned}
 \sum_{cyc} \frac{1}{a^2} &= \left(\sum_{cyc} \frac{1}{a} \right)^2 - 2 \sum_{cyc} \frac{1}{ab} = \\
 &= \left(\frac{ab+bc+ca}{abc} \right)^2 - \frac{2(a+b+c)}{abc} = \\
 &= \left(\frac{s^2+4Rr+r^2}{4Rrs} \right)^2 - \frac{4s}{4Rrs} = \left(\frac{s^2+4Rr+r^2}{4Rrs} \right)^2 - \frac{1}{Rr}
 \end{aligned}$$

and then

$$\begin{aligned}
 \sum_{cyc} \frac{1}{4R^2 \sin^2 A} &= \left(\frac{s^2+4Rr+r^2}{4Rrs} \right)^2 - \frac{1}{Rr} \iff \\
 \iff \sum_{cyc} \frac{1}{\sin^2 A} &= \frac{(s^2+4Rr+r^2)^2}{4r^2s^2} - \frac{4R}{r} \iff \\
 \sum_{cyc} (1 + \cot^2 A) &= \frac{(s^2+4Rr+r^2)^2}{4r^2s^2} - \frac{4R}{r} \iff \\
 \iff \sum_{cyc} \cot^2 A &= \frac{(s^2+4Rr+r^2)^2}{4r^2s^2} - 3 - \frac{4R}{r} = \\
 \frac{(s^2+4Rr+r^2)^2}{4r^2s^2} - 1 - \frac{4R}{r} - 2 &= \frac{(s^2-r^2-4Rr)^2}{4s^2r^2} - 2.
 \end{aligned}$$

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