p=a+b+c,q=ab+bc+ca,r=abc

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If $a,b,c>0$ and $abc=1$, then $p^2q^2+18pq-27\ge(p+q)^3$,

where $p:=a+b+c,\ q:=ab+bc+ca$.

Solution by Arkady Alt , San Jose, California, USA.

I will use mnemonically more convenient notation, namely $s:=a+b+c$ (s because sum), $p:=ab+bc+ca$ (p because pairly product) $q:=abc$.

In the such notations inequality of the problem becomes

(**1**) $s^2p^2+18sp-27\ge(s+p)^3$.

Vieta's system

(**V**) $\begin{cases} a+b+c=s \\ ab+bc+ca=p \\ abc=q \end{cases}$

solvable iff numbers s,p,q satisfy inequality*

(**B**) $p^2s^2-4p^3+18pqs-4qs^3-27q^2\ge 0$ (Sturm Theorem) [1].

Since $q=1$ then inequality becomes $p^2s^2-4p^3+18ps-4s^3-27\ge 0\iff$ $s^2p^2+18sp-27\ge 4(s^3+p^3)$. And also since $4(s^2-sp+p^2)\ge(s+p)^2\iff$ $3(p-s)^2\ge 0$ we have $4(s^3+p^3)\ge(s+p)^3$.(or, by Power Mean-Arithmetic Mean Inequality $\left(\dfrac{s^3+p^3}{2}\right)^{1/3}\ge\dfrac{s+p}{2}$). Hence, $s^2p^2+18sp-27\ge(s+p)^3$.

* **Proof of** (**B**).

Note that system (**V**) solvable in real numbers iff cubic equation $u^3-su^2+pu-q=0$ have three real solutions.

1. Proof (**with algebraic transformations**).

We will prove that cubical equation have three real roots a,b,c iff

$(a-b)^2(b-c)^2(c-a)^2\ge 0$.

Necessity.

If roots a,b,c of equation $u^3-su^2+pu-q=0$ are real then obvious that

$(a-b)^2(b-c)^2(c-a)^2\ge 0$.

Sufficiency.

Let $(a-b)^2(b-c)^2(c-a)^2\ge 0$ and suppose that a is real but b and c are complex numbers $b=\alpha+i\beta, c=\alpha-i\beta, \beta\ne 0$.

Then $(a-b)^2(b-c)^2(c-a)^2=((a-\alpha-i\beta)(a-\alpha+i\beta))^2(2\beta i)^2=$
$-4\beta^2\left((a-\alpha)^2+\beta^2\right)^2<0$. This contradict to $(a-b)^2(b-c)^2(c-a)^2\ge 0$.

Using identity

$(a-b)^2(b-c)^2(c-a)^2=(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)-$
$3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)$ and the following $s-p-q$ representations

$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)=p^2s^2-p^3-qs^3$,

$(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)=9q^2+p^3-6pqs+qs^3$

we obtain

$(a-b)^2(b-c)^2(c-a)^2=p^2s^2-4p^3+18pqs-4qs^3-27q^2$.

Thus desirable criteria is

$$p^2s^2 - 4p^3 + 18pqs - 4qs^3 - 27q^2 \geq 0.$$

1. D.S. Mitrinovic,J.E. Pecaric and V. Volenec, Recent Advances in Geometric Inequalities, p.6)