$$= 5x_nx_{n+1} + 1 - \left(x_n^2 + x_n + 1\right) = 5x_nx_{n+1} + 1 - x_{n-1}x_{n+1} =$$

$$= x_{n+1}\left(5x_n - x_{n-1}\right) + 1 = x_{n+1}\left(x_{n+1} + 1\right) + 1 = x_{n+1}^2 + x_{n+1} + 1$$

Using the Hölder's inequality we get

$$x_nx_mx_px_{n+2}x_{m+2}x_{p+2} =$$

$$= \left(x_{n+1}^2 + x_{n+1} + 1\right)\left(x_{m+1} + 1 + x_{m+1}^2\right)\left(1 + x_{p+1}^2 + x_{p+1}\right) \geq$$

$$\geq \left(\sqrt[3]{x_{n+1}^2 x_{m+1}} + \sqrt[3]{x_{p+1}^2 x_{n+1}} + \sqrt[3]{x_{m+1}^2 x_{p+1}}\right)^3$$

W31. (Solution by the proposer.) Using the Jensen's inequality we have

$$\left(\sum \log_{a_1} a_1a_2\right)^\lambda \geq n\left(\frac{1}{n}\sum \log_{a_1} a_1a_2\right)^\lambda = n\left(\frac{1}{n}\sum\left(\log_{a_1} a_1 + \log_{a_1} a_2\right)\right)^\lambda =$$

$$= n\left(\frac{1}{n}\left(n + \sum \log_{a_1} a_2\right)\right)^\lambda = n\left(1 + \sqrt[n]{\prod \log_{a_1} a_2}\right)^\lambda = n2^\lambda$$

Second solution. Let $x_k := \log_{a_k} a_{k+1}, k = 1, 2, ..., n-1$ and $x_n := \log_{a_{n+1}} a_1$. Then $x_k > 0, k = 1, 2, ..., n,$

$$\prod_{k=1}^{n} x_k = \prod_{cyc}^{n} \log_{a_1} a_2 = 1$$

and original inequality becomes

$$\sum_{k=1}^{n} (1 + x_k)^\lambda \geq n2^\lambda. \qquad (1)$$

I. First we will prove that for any $\lambda \in [0, \infty)$ inequality (1) holds.
Case $\lambda = 0$ is trivial. For any $\lambda \in (0, \infty)$ we have:

1. By AM-GM Inequality

$$\sum_{k=1}^{n} (1 + x_k)^\lambda \geq n\left(\prod_{k=1}^{n} (1 + x_k)^\lambda\right)^{\frac{1}{n}} = n\left(\prod_{k=1}^{n} (1 + x_k)\right)^{\frac{\lambda}{n}};$$

2. Also $1+x_k \geq 2\sqrt{x_k}$, $k=1,2,...,n$.
Hence,

$$\sum_{k=1}^{n}(1+x_k)^{\lambda} \geq n\left(\prod_{k=1}^{n}(1+x_k)\right)^{\frac{\lambda}{n}} \geq n\left(\prod_{k=1}^{n}2\sqrt{x_k}\right)^{\frac{\lambda}{n}} =$$

$$= n\left(2^n\sqrt{\prod_{k=1}^{n}x_k}\right)^{\frac{\lambda}{n}} = n\left(2^n\right)^{\frac{\lambda}{n}} = n2^{\lambda}.$$

II. Now consider case $\lambda < 0$.
We will show that for any such λ there are $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n > 0$ such that
$x_1x_2...x_n = 1$ *and*

$$\sum_{k=1}^{n}(1+x_k)^{\lambda} < n2^{\lambda} \iff \sum_{k=1}^{n}\frac{1}{(1+x_k)^p} < \frac{n}{2^p},$$

where $p := -\lambda > 0$.
For any $p > 0$ let $n > 2^p$and let $x_k = a > 0, k = 1,2,...,n-1$, $x_n = \dfrac{1}{a^{n-1}}$.
Then $1 < \dfrac{n}{2^p}$ and

$$\sum_{k=1}^{n}\frac{1}{(1+x_k)^p} = \frac{n-1}{(1+a)^p} + \frac{a^{(n-1)p}}{(1+a^{n-1})^p}.$$

Since

$$\lim_{a\to\infty}\left(\frac{n-1}{(1+a)^p} + \frac{a^{(n-1)p}}{(1+a^{n-1})^p}\right) = 1$$

then

$$\frac{n-1}{(1+a)^p} + \frac{a^{(n-1)p}}{(1+a^{n-1})^p} < \frac{n}{2^p}$$

for big enough a.
Simple particular case:
Let $n = 3$ and $\lambda = -1$ then for $x_1 = x_2 = 2, x_3 = 1/4$ we obtain

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}=\frac{2}{1+2}+\frac{1}{1+1/4}=\frac{22}{15}<\frac{3}{2}=3\cdot 2^{\gamma}.$$

Arkady Alt

W32. (Solution by the proposer.) The function

$$f(x)=x\ln(1-x)$$

is concave because

$$f''(x)=-\frac{1}{1-x}-\frac{1}{(1-x)^2}<0$$

Using the Popoviciu's inequality

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right)\le 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right)$$

we obtain the desired result.

Second solution. Note that

$$8\prod_{cyc}(1-x)^x\le 3\prod_{cyc}(2-y-z)^{y+z}\iff 8\prod_{cyc}(1-x)^x\le$$

$$\le 3\prod_{cyc}(1+x)^{1-x}\iff$$

$$\iff\prod_{cyc}\left(1-x^2\right)^x\le\frac{3}{8}\prod_{cyc}(1+x).$$

Since by AM-GM Inequality

$$\prod_{cyc}\left(1-x^2\right)^x\le\sum_{cyc}x\left(1-x^2\right)=1-\left(x^3+y^3+z^3\right)$$

suffices to prove inequality

$$1-\left(x^3+y^3+z^3\right)\le\frac{3}{8}(1+x)(1+y)(1+z).\qquad(1)$$

Let $p:=xy+yz+zx, q:=xyz$. Then

$$x^3+y^3+z^3=1+3q-3p,$$