**W27. (Solution by the proposer.)** By the AM-GM inequality

$$a_1 + \frac{1}{\prod_{j=1}^{k} a_j} \geq 2\sqrt{\frac{1}{\prod_{j=1}^{k} a_j}} = \frac{2}{\sqrt{\prod_{j=2}^{k} a_j}}$$

therefore the left-hand side of the proposed inequality is

$$LHS \geq \sqrt[n]{2} \sum_{cyclic} \frac{1}{\sqrt[2n]{\prod_{j=1}^{k-1} a_j}}.$$

Now, since function $f(x) = \frac{1}{\sqrt[2n]{x}}$ is convex for $x > 0$ the conclusion follows by Jensen inequality.

**Second solution.** Let $A := \prod_{j=1}^{k} a_j$ and $b_i := \frac{A}{a_i}, i = 1, 2, .., k$. Since

$$\sum_{cyc} \prod_{j=1}^{k-1} a_j = k \iff \sum_{i=1}^{k} b_i = k$$

and

$$\prod_{i=1}^{k} b_i = \prod_{i=1}^{k} \frac{A}{a_i} = \frac{A^k}{\prod_{j=1}^{k} a_j} = A^{k-1}$$

by AM-GM Inequality we obtain

$$k = \sum_{i=1}^{k} b_i \geq k \sqrt[k]{\prod_{i=1}^{k} b_i} = k \sqrt[k]{A^{k-1}} \iff A \leq 1.$$

From the other hand

$$\sum_{cyc} \sqrt[n]{a_1 + \frac{1}{\prod_{j=1}^{k} a_j}} = \sum_{i=1}^{k} \sqrt[n]{a_i + \frac{1}{A}} \geq$$

$$\geq \sum_{i=1}^{k} \sqrt[n]{a_i + \frac{1}{A}} \geq \sum_{i=1}^{k} \sqrt[n]{2\sqrt{\frac{a_i}{A}}} = \sum_{i=1}^{k} \sqrt[n]{\frac{2}{\sqrt{b_i}}} = \sqrt[n]{2} \sum_{i=1}^{k} \frac{1}{\sqrt[2n]{b_i}}.$$

And again applying AM-GM Inequality we obtain

$$\sum_{i=1}^{k} \frac{1}{\sqrt[2n]{b_i}} \geq k \sqrt[k]{\prod_{j=1}^{k} \frac{1}{\sqrt[2n]{b_i}}} = k \sqrt[2nk]{\frac{1}{\prod_{j=1}^{k} b_i}} = k \sqrt[2nk]{\frac{1}{A^{k-1}}} \geq k$$

because $A \leq 1$. Thus,

$$\sum_{cyc} \sqrt[n]{a_1 + \frac{1}{\prod_{j=1}^{k} a_j}} \geq k \sqrt[n]{2}.$$

Arkady Alt

**W28.** (**Solution by the proposer.**) The roots of characteristic polynomial equation $x^2 - x + \frac{1}{2} = 0$ are $\alpha = \frac{1+i}{2}$ and $\beta = \frac{1-i}{2}$, and taking into account the initial values of the sequence, it follows that $x_n = \alpha^n + \beta^n$, so the proposed series reads may be written as

$$\sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{n+2}.$$

The generating function for $(x_n)_{n \geq 0}$ is $F(x) = \sum_{n=0}^{\infty} x_n x^n = \frac{2-x}{\frac{x^2}{2} - x + 1}$.

Funtion $F(z)$ is analytic for $z \in C$ with $|z| < \sqrt{2}$.

To find the proposed series $\sum_{n=1}^{\infty} \frac{x_n}{n+2} = \sum_{n=1}^{\infty} \frac{\alpha^n + \beta^n}{n+2}$, we consider the function $x \cdot F(x) = \frac{2x - x^2}{\frac{x^2}{2} - x + 1}$, and its integral

$$\int \frac{2x - x^2}{\frac{x^2}{2} - x + 1} dx = -2\left(x + 2\tan^{-1}(1-x)\right) = G(x)$$

so $\sum_{n=1}^{\infty} \frac{x_n x^{n+2}}{n+2} = G(x) - G(0) = \pi - 2\left(x + 2\tan^{-1}(1-x)\right)$.

Therefore, $\sum_{n=1}^{\infty} \frac{x_n}{n+2} = G(1) - G(0) = \pi - 2\left(1 + 2\tan^{-1}(1)\right) = \pi - 2$.

**Second solution.** Note that

$$x_{n+2} = x_{n+1} - \frac{1}{2} x_n \iff$$