$$x = y\left(\ln y + \ln\ln y + o\left(\ln\ln y\right)\right)$$

we deduce

$$p_n = n\left(\ln n + \ln\ln n + o\left(\ln\ln n\right)\right) \qquad (*)$$

for $n \to +\infty$

$$\left|\frac{e^{i\ln(p_n)}}{p_n} - (n\ln n)^{i-1}\right| \le \sqrt{2}\frac{1}{n^2\ln^2 n}\left|p_n - n\ln n\right|$$

with (*) we get

$$\frac{e^{i\ln(p_n)}}{p_n} - (n\ln n)^{i-1} = O_{n\to+\infty}\left(\frac{\ln\ln n}{n\ln^2 n}\right)$$

since $\sum_n \frac{\ln\ln n}{n\ln^2 n}$ converge, $\sum\limits_{n>0} \frac{e^{i\ln(p_n)}}{p_n}$ and $\sum\limits_{n>0} (n\ln n)^{i-1}$ are same nature. With integral and series we have

$$\int_2^{n+1} (t\ln t)^{i-1}\,dt = \int_2^{n} (t\ln t)^{i-1}\,dt + (n\ln n)^{i-1} + v_n$$

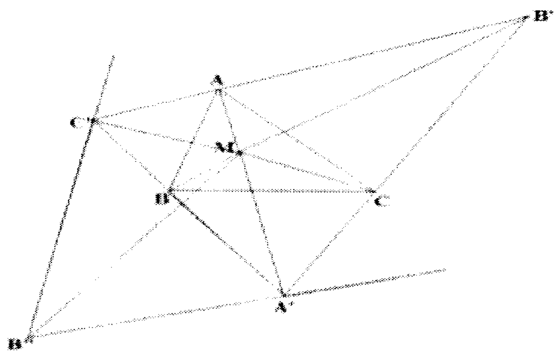
with $|v_n| \le \frac{K}{n^2\ln n}$, then

$$\sum_{n=2}^{N} (n\ln n)^{i-1} = \int_2^{n+1} (t\ln t)^{i-1}\,dt = \sum_{n=2}^{N} v_n$$

has a finite limit when $n \to \infty$. This prove $\sum\limits_{n\ge1} \frac{e^{i\ln(p_n)}}{p_n}$ converge

W26. (Solution by the proposer.) We constructing the perpendiculars on MA, MB, MC in the points A, B, C. These perpendiculars meet in the points A', B', C', see the figure. In the same way, in the points A', B', C' we construct the perpendiculars on MA', MB', MC' which meet in A'', B'', C''.

We calculate the area of the triangle $MA'C'$ in two ways thus:



$$\frac{MA' \cdot MC' \cdot \sin A'MC'}{2} = \frac{MB \cdot A'C'}{2} = \frac{MB \cdot MB'' \cdot \sin A'MC'}{2}$$

So, we deduce

$$MB'' = \frac{MA' \cdot MC''}{MB}$$

Similarly, we obtain:

$$MA'' = \frac{MB' \cdot MC'}{MA} \cdot MC'' = \frac{MA' \cdot MB'}{MC}$$

From Erdős-Mordell's inequality applied in the triangle $A''B''C''$ for the point M, we have:

$$MA'' + MB'' + MC'' \geq 2\left(MA' + MB' + MC'\right)$$

so

$$\frac{MB' \cdot MC'}{MA} + \frac{MA' \cdot MC'}{MB} + \frac{MB' \cdot MA'}{MC} \geq 2\left(MA' + MB' + MC'\right)$$

Since $MA' = 2R_a, MB' = 2R_b, MC' = 2R_c$, it follows:

$$\frac{R_bR_c}{MA} + \frac{R_aR_c}{MB} + \frac{R_aR_b}{MC} \geq R_a + R_b + Rc$$

Therefore, the inequality of the statement.

Second solution. We will use for the radii of circumcircle of MBC, MCA, MAB another notation ρ_a, ρ_b, ρ_c and standard notation R_a, R_b, R_c for distances MA, MB, MC, respectively.

Also denote via d_a, d_b, d_c distances from M to BC, CA, AB.

Thus, inequality to prove is

$$\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\le\frac{1}{R_a}+\frac{1}{R_b}+\frac{1}{R_c}. \tag{1}$$

Since $[BMC]=\dfrac{ad_a}{2}$ and

$$4\rho_a\,[BMC]=aR_bR_c \iff 2\rho_a\cdot ad_a=aR_bR_c \iff \frac{1}{\rho_a}=\frac{2d_a}{R_bR_c}$$

then (1) $\iff \sum\limits_{cyc}\dfrac{2d_a}{R_bR_c}\le\sum\limits_{cyc}\dfrac{1}{R_a}.$

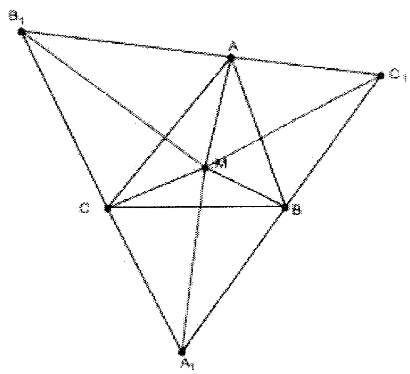


Figure 1

Let us draw throughout vertexes A,B,C respectively three lines perpendicularly to MA,MB,MC respectively. Three points of intersection of these lines determine triangle $A_1B_1C_1$
($A_1B_1\perp MC,\ B_1C_1\perp MA,\ C_1A_1\perp MB$) with
$d_{a_1}:=R_a=MA, d_{b_1}:=R_b=MB, d_{c_1}:=R_c=MC$ as distances from M to B_1C_1,C_1A_1,A_1B_1 respectively and with distances between M and vertexes of $\triangle A_1B_1C_1$: $R_{a_1}:=MA_1, R_{b_1}:=MB_1, R_{c_1}:=MC_1$.(Pic.1).
Since R_{a_1},R_{b_1},R_{c_1} are diameters of the circumcircles for quadrilaterals $MCA_1B,\ MAB_1C,\ MBC_1A$,respectively, then, by Sine-Theorem, we have

$$R_{a_1}=\frac{a}{\sin\angle BMC}=\frac{aR_bR_c}{R_bR_c\sin\angle BMC}=\frac{aR_bR_c}{ad_a}=\frac{R_bR_c}{d_a}$$

and,similarly,

$$R_{b_1}=\frac{R_cR_a}{d_b},R_{c_1}=\frac{R_aR_b}{d_c}.$$

Since

$$\sum_{cyc} \frac{2d_a}{R_b R_c} = 2\sum_{cyc} \frac{1}{R_{a_1}}$$

and

$$\sum_{cyc} \frac{1}{R_a} = \sum_{cyc} \frac{1}{d_{a_1}}$$

then **(1)** $\iff 2\sum_{cyc} \frac{1}{R_{a_1}} \le \sum_{cyc} \frac{1}{d_{a_1}}$.
Thus, suffices to prove that in any triangle ABC with interior point M and R_a, R_b, R_c as distances from point M to vertices A, B, C,respectively, and d_a, d_b, d_c distances from M to BC, CA, AB holds inequality

$$2\sum_{cyc} \frac{1}{R_a} \le \sum_{cyc} \frac{1}{d_a}. \tag{2}$$

Lemma. Let P_a and P_b be involutions of P with respect to a and b respectively (that is $PP_a \perp a$, $PP_b \perp b$, $P_aM \cdot PM = P_bN \cdot PN = 1$)
Then $P_aP_b \perp PA$ and $PE = \dfrac{1}{PA}$ where E is intersection point of P_aP_b and PA.
Proof.

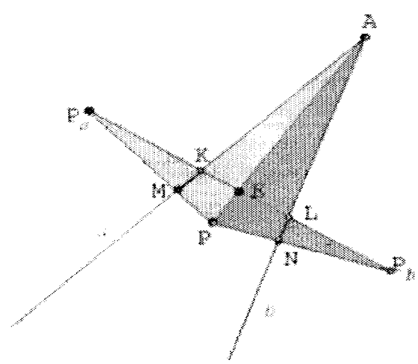


Figure 2

Let P_aE_1 and P_bE_2 be perpendiculars from P_a and P_b to $\overleftrightarrow{PA}$ respectively $(E_1, E_2 \in \overleftrightarrow{PA}$). Since $\angle PP_aE_1 = \angle PAM$ and $\angle PP_bE_2 = \angle PAN$ (as the angles which constructed by mutually perpendicular sides) then we have $\triangle PP_aE_1 \sim \triangle PAM$ and $\triangle PP_bE_2 \sim \ \triangle PAN$ and from these similarity follows

$$\frac{PE_1}{PP_a} = \frac{PM}{PA} \iff \frac{PE_1}{\frac{1}{d_a}} = \frac{d_a}{PA} \iff PE_1 = \frac{1}{PA}$$

and

$$\frac{PE_2}{PP_b} = \frac{PN}{PA} \iff \frac{PE_2}{\frac{1}{d_b}} = \frac{d_b}{PA} \iff PE_2 = \frac{1}{PA}.$$

Hence, $PE_1 = PE_2$ and $E := E_1 = E_2$ is intersection point of P_aP_b with PA and $PE = \dfrac{1}{R_A}$.

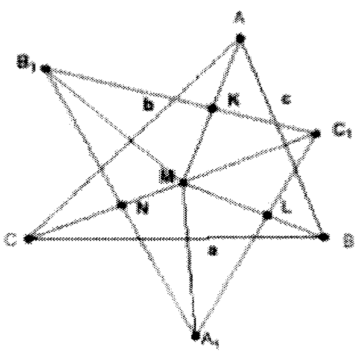


Figure 3

Let A_1, B_1, C_1 be involution points for M with respect to lines $\overleftrightarrow{BC}, \overleftrightarrow{CA}, \overleftrightarrow{AB}$ respectively. Let $R_a^{'} = MA_1 = \dfrac{1}{d_a}, R_b^{'} = MB_1 = \dfrac{1}{d_b}, R_c^{'} = MC_1 = \dfrac{1}{d_c}$ and $d_a^{'}, d_b^{'}, d_c^{'}$ be distances from M to sides B_1C_1, C_1A_1, A_1B_1.Since by lemma $d_a^{'} = \dfrac{1}{R_a}, d_b^{'} = \dfrac{1}{R_b}, d_c^{'} = \dfrac{1}{R_c}$ then replacing $(R_a, R_b, R_c, d_a, d_b, d_c)$ in Erdös-Mordell Inequality$R_a + R_b + R_c \geq 2(d_a + d_b + d_c)$
with

$$\left(R_a^{'}, R_b^{'}, R_c^{'}, d_a^{'}, d_b^{'}, d_c^{'}\right)$$

we obtain

$$\sum_{cyc} R_a^{'} \geq 2\sum_{cyc} d_a^{'} \iff (2)$$

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