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Let $a > b$ be positive real numbers and let n be a positive integer. Prove that

$$\frac{(a^{n+1}-b^{n+1})^{n-1}}{(a^n-b^n)^n} > \frac{n}{(n+1)^2} \cdot \frac{e}{a-b}$$

where e is the Euler number.

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Noting that $\frac{(a^{n+1}-b^{n+1})^{n-1}}{(a^n-b^n)^n} > \frac{n}{(n+1)^2} \cdot \frac{e}{a-b} \Leftrightarrow \frac{(a-b)(a^{n+1}-b^{n+1})^{n-1}}{(a^n-b^n)^n} > \frac{en}{(n+1)^2} \Leftrightarrow$ $\frac{(1-t)(1-t^{n+1})^{n-1}}{(1-t^n)^n} > \frac{en}{(n+1)^2}$, where $t := \frac{b}{a} \in (0,1)$ and $e < \left(1+\frac{1}{n}\right)^{n+1} = \frac{(n+1)^{n+1}}{n^{n+1}}$

we will prove more stronger inequality

(**1**) $\frac{(1-t)(1-t^{n+1})^{n-1}}{(1-t^n)^n} \geq \frac{(n+1)^{n-1}}{n^n}$

which yields original. Let $p_n := \frac{1+t+\ldots+t^{n-1}}{n}, n \in \mathbb{N}$. Since $\frac{(1-t)(1-t^{n+1})^{n-1}}{(1-t^n)^n} =$ $\frac{(1-t)^n(1+t+\ldots+t^n)^{n-1}}{(1-t^n)^n} = \frac{(1+t+\ldots+t^n)^{n-1}}{(1+t+\ldots+t^{n-1})^n}$ then inequality (**1**) becomes

$$\frac{((n+1)p_{n+1})^{n-1}}{(np_n)^n} \geq \frac{(n+1)^{n-1}}{n^n} \Leftrightarrow p_{n+1}^{n-1} \geq p_n^n \Leftrightarrow \sqrt[n]{p_{n+1}} \geq \sqrt[n-1]{p_n}.$$

Lemma 1.

If $t > 0$ then $(p_n)_{n\in\mathbb{N}}$ is Log-Concave sequence, namely for any $n \geq 2$ holds inequality

(**2**) $p_{n+1} \cdot p_{n-1} \geq p_n^2$

Proof.

If $t = 1$ then $p_n = 1, n \in \mathbb{N}$ and inequality (2) obviously holds. Let $t \neq 1$. Since $p_n = \frac{1-t^n}{(1-t)n}$ then

(**2**)$\Leftrightarrow \frac{1-t^{n+1}}{n+1} \cdot \frac{1-t^{n-1}}{n-1} \geq \left(\frac{1-t^n}{n}\right)^2 \Leftrightarrow n^2(1-t^{n-1}-t^{n+1}+t^{2n}) \geq (n^2-1)(1-2t^n+t^2) \Leftrightarrow$ $(1-t^n)^2 \geq n^2t^{n-1}(1-t)^2 \Leftrightarrow 1-t^n \geq nt^{\frac{n-1}{2}}(1-t) \Leftrightarrow \frac{1+t+\ldots+t^{n-1}}{n} \geq nt^{\frac{n-1}{2}}$,

where latter inequality is right because by AM-GM we have

$$\frac{1+t+\ldots+t^{n-1}}{n} \geq \sqrt[n]{1\cdot t\cdot\ldots\cdot t^{n-1}} = \sqrt[n]{t^{1+2+\ldots+n-1}} = \sqrt[n]{t^{\frac{(n-1)n}{2}}} = t^{\frac{n-1}{2}}.$$

Lemma 2.

For any positive integer $n \geq 2$ and any positive real t holds inequality

(**3**) $\left(\frac{1+t+\ldots+t^n}{n+1}\right)^{\frac{1}{n}} \geq \left(\frac{1+t+\ldots+t^{n-1}}{n}\right)^{\frac{1}{n-1}} \Leftrightarrow \sqrt[n]{p_{n+1}} \geq \sqrt[n-1]{p_n}.$

Proof.

Noting that (**3**)$\Leftrightarrow p_{n+1}^{n-1} \geq p_n^n$ we will prove latter inequality by Math Induction.

1.Base of Math Induction.

Let $n = 2$ then $p_{n+1}^{n-1} \geq p_n^n$ becomes $p_3 \geq p_2^2 \Leftrightarrow \frac{1+t+t^2}{3} \geq \frac{(1+t)^2}{4} \Leftrightarrow (t-1)^2 \geq 0.$

2. Step of Math Induction.

Assume that for any $n \geq 2$ holds inequality $p_{n+1}^{n-1} \geq p_n^n$. Since (Lemma 1) $p_{n+2} \cdot p_n \geq p_{n+1}^2$

for any $n \geq 1$ then $\left(p_{n+2} \cdot p_n\right)^n \geq p_{n+1}^{2n\text{‘}} \Leftrightarrow \frac{p_{n+2}^n}{p_{n+1}^{n-1}} \geq \frac{p_{n+1}^{n+1\text{‘}}}{p_n^n}$ and, therefore,

$p_{n+2}^n = \frac{p_{n+2}^n}{p_{n+1}^{n-1}} \cdot p_{n+1}^{n-1} \geq \frac{p_{n+1}^{n+1\text{‘}}}{p_n^n} \cdot p_n^n = p_{n+1}^{n+1\text{‘}}$.