https://www.linkedin.com/feed/update/urn:li:activity:6475308720208388096

TOTTEN-11. **Proposed by Wolter Janous**, **Ursulinengimnasium**, **Insbruck**,**Austria**.

(a) Let $x, y,$ and z be positive real numbers such that $x + y + z = 1.$ Prove that

$$\frac{8\sqrt{3}}{9} \leq \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)\left(\frac{1}{\sqrt{z}} - \sqrt{z}\right).$$

(b)★ Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \ldots + x_n = 1.$ Prove or disprove that

$$\left(\frac{n-1}{\sqrt{n}}\right)^n \leq \prod_{k=1}^{n}\left(\left(\frac{1}{\sqrt{x_k}} - \sqrt{x_k}\right)\right).$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

(a) Solution 1.(Reduction to geometric inequality).

Using notation $a := 1 - x, b := 1 - y, c := 1 - z$ we obtain for the right hand side of original inequality following representation

$$\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)\left(\frac{1}{\sqrt{z}} - \sqrt{z}\right) = \frac{(1-x)(1-y)(1-z)}{\sqrt{xyz}} = \frac{abc}{F},$$

where F is area of triangle determined by sides with lengths $a, b,$ and c and semiperimeter $s = 1$ ($a + b + c = 2$ and for positive a, b, c holds inequalities $a < 1 \Leftrightarrow a < b + c, b < 1 \Leftrightarrow b < c + a, c < 1 \Leftrightarrow c < a + b$).

Thus, original inequality in such geometric interpretation and in homogeneous form is

(1) $\quad \dfrac{8\sqrt{3}}{9} \leq \dfrac{abc}{sF}.$

Let R be circumradius of this triangle, then $\dfrac{abc}{sF} = \dfrac{4FR}{sF} = \dfrac{4R}{s}$ and, therefore, (1)$\Leftrightarrow$

(2) $\quad a + b + c \leq 3\sqrt{3}R.$

(2) is well known inequality and can be proved, for example, by application Sine-Theorm, in form of the folloving trigonometric inequality

(3) $\quad \sin A + \sin B + \sin C \leq \dfrac{3\sqrt{3}}{3}$

which immediately follows from application Jensen's Inequality for concave down on $[0, \pi]$ function $\sin x$.

Solution 2.(Direct algebraic proof).

Let $p := xy + yz + zx$ and $q := xyz$ then original inequality in (a) becomes

(4) $\quad \dfrac{8\sqrt{3}}{9} \leq \dfrac{p-q}{\sqrt{q}}.$

Since $\dfrac{p-q}{\sqrt{q}} = \dfrac{p}{\sqrt{q}} - \sqrt{q}$ obviously decreasing in q on $(0, \infty)$ and

$3p \leq 1(\Leftrightarrow 3(xy + yz + zx) \leq (x + y + z)^2),$

$3q \leq p^2$ $(\Leftrightarrow 3xyz(x + y + z) \leq (xy + yz + zx)^2)$

then $\dfrac{p-q}{\sqrt{q}} \geq \dfrac{p}{\sqrt{\dfrac{p^2}{3}}} - \sqrt{\dfrac{p^2}{3}} = \sqrt{3} - \dfrac{p}{\sqrt{3}} = \dfrac{3-p}{\sqrt{3}}$ and, therefore,

$$\frac{p-q}{\sqrt{q}} - \frac{8\sqrt{3}}{9} = \left(\frac{p-q}{\sqrt{q}} - \frac{3-p}{\sqrt{3}}\right) + \frac{3-p}{\sqrt{3}} - \frac{8\sqrt{3}}{9} =$$
$$\left(\frac{p-q}{\sqrt{q}} - \frac{3-p}{\sqrt{3}}\right) + \frac{1-3p}{3\sqrt{3}} \ge 0.$$

(b).

Inequality incorrect for $n = 2$.

Indeed, since $x_1 + x_2 = 1$ then $\frac{(1-x_1)(1-x_2)}{\sqrt{x_1x_2}} = \frac{x_2x_1}{\sqrt{x_1x_2}} = \sqrt{x_1x_2}$

and inequality in (b) becomes $\left(\frac{1}{\sqrt{2}}\right)^2 \le \frac{(1-x_1)(1-x_2)}{\sqrt{x_1x_2}} \Leftrightarrow \frac{1}{2} \le \sqrt{x_1x_2}$.

But $\sqrt{x_1x_2} \le \frac{x_1+x_2}{2} = \frac{1}{2}$.